BIOGRAPHIES AND COLLECTED WORKS OF MATHEMATICIANS

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"No one should be recognized a master in any subject who does not know at least the outline of its history. Of course it would be foolish to expect him to have any deep historical knowledge, but he should know the main landmarks and the leading personalities—he should be acquainted with his scientific ancestors.

"This is almost a moral obligation. We might compare it to the obligation for any educated citizen to know the history of his country. The obligation is of the same kind and of the same order. . . . For a physicist not to be sufficiently familiar with Galileo and Newton is just as shocking as for an American not to know Washington and Lincoln."

These are the words of George Sarton,* historian of science. A decade ago they motivated the present writer to take up what has proved to be a most satisfying and profitable avocation: to seek out, to read, and to record:

I. All *book-length* biographies (individual and collected) in English of physicists, astronomers, mathematicians, chemists, metallurgists and engineers.

II. All of the more important writings in English on the historical development of electrophysics and electrical engineering and on the lives of noted electrophysicists and electrical engineers.

Literal accomplishment of this program is, of course, virtually impossible: but it can be carried to a high degree of completion. And now that the hunt for material has encompassed search of (i) complete files of the consequential serial publications in English devoted to physics or to electrical engineering; (ii) the stacks and the card catalogs of many of the important public, university and technical libraries located in the East and Middle West; (iii) the accumulated catalogs of the principal publishers of technical and scientific books; (iv) the lists of offerings, over a decade, of the larger dealers in used and rare technical and scientific works; (v) much relevant miscellaneous bibliographical material—book review journals, printed catalogs of private libraries, and kindred aids—it is believed that much the greater part of the more worthwhile material has been located and read. In such thought it seems desirable that this material now be made available to those similarly interested.

At present the file devoted to items of category I contains some 800 titles; that devoted to category II, some 1500 titles. The first part of the bibliography below comprises the items of I apposite to mathematicians. Elsewhere are to appear separate bibliographies on physicists and astronomers,† on chemists,‡ and on metallurgists and engineers;§ and the complete bibliography of some

* Sarton, G., The History of Science and the New Humanism, Harvard University Press, Cambridge, 1936, pp. 43–44.

† Higgins, T. J., Book-length biographies of physicists and astronomers, American Journal of Physics, vol. 12, 1944, p. 31–39; addendum, vol. 12, 1944, pp. 234–236.

‡ Higgins, T. J., Book-length biographies of chemists, School Science and Mathematics, vol. 44, 1944, pp. 650–665.

§ Higgins, T. J., Book-length biographies of engineers and metallurgists, Bulletin of Bibliography, forthcoming.

1500 titles on the historical development of electrophysics and electrical engineering and on the lives and work of the more noted of those who labored in these domains.

The writer will not attempt here to discuss what is to be gained through reading one, several, or many of the titles listed. His own views on the use of biographical material as a tool in teaching have been presented elsewhere.* Again, in several books†,‡ and in numerous papers published in Isis and in other journals, Dr. Sarton had discussed in detail the special values of the study of the history of science, technology and mathematics, and of the use of biographical material therein. Of unusual interest—with respect to the present paper—is his argument:‡

"The main reason for studying the history of mathematics, or the history of any science, is purely humanistic. Being men, we are interested in other men, and especially in such men as have helped us to fulfill our highest destiny. As soon as we realize the great part played by individual men in mathematical discoveries—for, however these may be determined, they cannot be brought about except by means of human brains—, we are anxious to know all their circumstances. . . . All these questions and many others are deeply interesting, especially for other mathematicians: if they are young, because of their dreams of the future and their hopes and doubts; if they are older, because of their memories of the past, and also, though in a different way because of their hopes and doubts."

Finally, we may note a fact remarked by many writers; their discussions may be taken as epitomized in the words of that noted British historian of engineering H. W. Dickinson,§ authoritative biographer of a half-dozen pioneer engineers of note:

"To the young person especially, but also to the old, the study of biography is valuable, as it broadens the mind, quickens the imagination, fires ambition and strengthens character. One thing the young man will find that will encourage him—and there is a period in everyone's life when encouragement is needed—and that is, that in one respect youth is at the helm, for practically all [mathematical!] inventions are made between the ages of twenty and forty. . . ."

The second part of the bibliography below comprises a list of the available "collected works" in English of those who worked in either (occasionally, in both) pure or applied mathematics. If biographical memoirs (a term we use as a collective for the various phrases: biographical sketch, personal recollections, obituary, *etc.*) are to be found in a "collected works," the authors thereof are

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* Higgins, T. J., The function of biography in engineering education, Journal of Engineering Education, vol. 32, 1941, pp. 82–92.

† Sarton, G., The Study of the History of Science, Harvard University Press, Cambridge, 1926, 75 pp.

‡ Sarton, G., The Study of the History of Mathematics, Harvard University Press, Cambridge, 1936, 113 pp.

§ Dickinson, H. W., The Value of History in Engineering Education, Rensselaer Polytechnic Institute Bulletin No. 55.

stated. Commonly, these men were intimates of the individuals about whom they wrote; for which reason it not unoften happens that these memoirs comprise the best existing account of the life and work of the individual in question.

This benefaction, however, is incidental to the major reason for setting forth the bibliography of "collected works" in English—which reason has been excellently stated by Dr. Sarton:

"The intelligent reading of the life of a mathematician would naturally lead to his works. Students may thus be induced to attack one of his treatises, or to examine a whole series of his papers. For this purpose my list will be useful, as it will tell them quickly whether 'collected works' are available or not. The existence of 'collected works' makes the study of the evolution of a man's thought much easier; without them one is obliged to refer to odd volumes of a number of periodicals, a process very tiresome at best, and for those out of reach of a large library, almost impossible.

. . .

"A student owning a good biography and the collected works of a mathematician can begin and continue at leisure a deep study of his thought and fate. The reading of monographs will give him a more intimate contact with the realities and creative processes of mathematics than almost any textbook."

And anticipating the purist who surely will decry the worth of such study—"The pursuit of deceased mathematics," to quote one belittler of this phase of the study of the history of mathematics—we may well conclude with the words of that eminent divine, Bishop William Stubbs, who, *re* the value of history in general, wrote:

"The roots of the present lie deep in the past and nothing in the past is dead to the man who would learn how the present comes to be what it is."

INDIVIDUAL BIOGRAPHIES*

John Arbuthnot, Mathematician and Satirist. By L. M. Beattie. Cambridge, Harvard University Press, 1935. 432 pp.

Archimedes. By T. L. Heath. London, Society for Promoting Christian Knowledge; New York, The Macmillan Company, 1920. 58 pp.

Passages from the Life of a Philosopher. By C. Babbage. London, Longman, Green, Longman, Roberts and Green, 1864. 496 pp.

Roger Boscovich, S. J., 1711–1787. By H. V. Gill. Dublin, M. H. Gill and Son, Ltd., 1941. 76 pp.

Yankee Stargazer; the Life of Nathaniel Bowditch. By R. E. Berry. London and New York, Whittlesey House, McGraw-Hill Book Company, 1941. 234 pp.

Memoir of Nathaniel Bowditch by His Son Nathaniel Ingersoll Bowditch. Third Edition. By N. I. Bowditch. Boston, C. C. Little and J. Brown, 1840. 172 pp.; Cambridge, University Press, 1884. 178 pp.

* For the most part this section and that of COLLECTED BIOGRAPHIES encompass only those who worked wholly in, or made substantial contributions to, pure mathematics. For biographies of those who worked in applied mathematics (*i.e.*, mathematical physicists) see footnotes p. 433.

Eulogy on Nathaniel Bowditch, LL.D., President of the American Academy of Arts and Sciences; Including an Analysis of His Scientific Publications. By J. Pickering. Boston, C. C. Little and J. Brown, 1838. 101 pp.

Navigator; the Story of Nathaniel Bowditch. By A. Stanford. New York, William Morrow and Company, Inc., 1927. 308 pp.

An Eulogy on the Life and Character of Nathaniel Bowditch. By D. A. White. Salem, Mass., Printed at the Office of the Gazette, 1838. 72 pp.

A Discourse on the Life and Character of the Hon. Nathaniel Bowditch, LL.D., F.R.S. By A. Young. Boston, C. C. Little and J. Brown, 1838. 119 pp.

The Book of My Life (De Vita Propria Liber). By J. Cardan. Translated by J. Stoner. New York, E. P. Dutton and Company, 1930. 331 pp.

Jerome Cardan, The Life of Gerolamo Cardano of Milan, Physician. 2 Volumes. London, Chapman and Hall, Ltd., 1854.

Jerome Cardan, A Biographical Study. By W. G. Waters. London, Lawrence and Bullen, Ltd., 1898. 301 pp.

A Memoir of Zerah Colburn; Written by Himself. By Z. Colburn. Springfield, Mass., G. and C. Merriam, 1833. 204 pp.

Descartes: His Life and Times. By E. S. Haldane. New York, E. P. Dutton and Company, 1925. 398 pp.

Descartes. By J. P. Mahaffy. Edinburgh, W. Blackwood and Sons; Philadelphia, J. B. Lippincott and Company, 1881. 211 pp.

F. X. Edgeworth's Contributions to Mathematical Statistics. By A. L. Bowley. London, Royal Statistical Society, 1928. 139 pp. Contains no biographical details; an analysis of his mathematical work.

Carl Friedrich Gauss. By G. W. Dunnington. Baton Rouge, Louisiana State Univ. Press, 1937. 91 pp.

Willard Gibbs. By M. Rukeyser. New York, Doubleday, Doran and Company, Inc., 1942. 465 pp.

James Gregory. Tercentenary Memorial Volume. Containing His Correspondence with John Collins and His Hitherto Unpublished Mathematical Manuscripts, Together with Addresses and Essays Communicated to The Royal Society of Edinburgh, July 4, 1938. Edited by H. W. Turnbull for the Royal Society of Edinburgh. London, G. Bell and Sons, Ltd., 1939. 524 pp. This work contains an interesting chapter, Mathematicians of the Seventeenth Century, which encompasses brief sketches of the life and work of Collins, Wright, Briggs, Harriot, Oughtred, and Warner.

Life of Sir William Rowan Hamilton, Knt., LL.D., D.C.L., M.R.I.A., Andrews Professor of Astronomy in the University of Dublin, and Royal Astronomer of Ireland, etc.: Including Selections from His Poems, Correspondence and Miscellaneous Writings. By R. P. Graves. 3 Volumes. Dublin, Hodges, Figgis, and Company, 1882–89.

A Mathematician's Apology. By G. H. Hardy. Cambridge, University Press, 1940. 93 pp.

Thomas Hariot, the Mathematician, the Philosopher and the Scholar, Developed Chiefly from Dormant Materials. By H. Stevens. London, privately printed, 1900. 214 pp.

The Chequered Career of Ferdinand Rudolph Hassler, First Superintendent of the United States Coast Survey. By F. Cajori. Boston, The Christopher Publishing House, 1929. 245 pp.

Letters and Journals of W. Stanley Jevons. By W. S. Jevons. Edited by his wife. London, Macmillan and Company, 1886. 473 pp.

Sónya Kovalévsky, a Biography by Anna Carlotta Leffler, Duchess of Cajanello—translated by A. de Furuhjelm and A. M. Clive Bayley; and *The Sisters Rajevsky, Being an Account of Her Life by Sonya Kovalevsky*—translated by A. M. Clive Bayley. London, T. Fisher Unwin, Ltd., 1895. 377 pp. Contains a memoir of the Duchess of Cajanello by L. Wolffsohn.

Sonia Kovalévsky; Biography and Autobiography. I. Memoir, by *A. C. Leffler (Edgren) Duchessa di Cajanello; II. Reminiscences of Childhood Written by Herself*—translated by L. von Cossel. London, W. Scott, Ltd., 1895. 317 pp.

Sónya Kovalévsky; Her Recollections of Childhood—translated by I. F. Hapgood; *with Biography by Anna Carlotta Leffler, Duchess of Cajanello*—translated by A. M. Clive Bayley. New York, Century Company, 1895. 318 pp. Contains a memoir of the Duchess of Cajanello by L. Wolffsohn.

Memoirs of John Napier of Merchiston, His Lineage, Life, and Times. By M. Napier. Edinburgh, W. Blackwood, 1834. 534 pp.

An Address on the Genius and Discoveries of Sir Isaac Newton. By G. Boole. Lincoln, Gazette Office, 1835. 23 pp.

The Life of Sir Isaac Newton. By D. Brewster. London, John Murray. 366 pp.; New York, J. and J. Harper, 1831, 1835; New York, Harper and Brothers, 1843. 321 pp. Revised and edited by W. T. Lynn. London, W. Tegg and Company, 1875. 346 pp. The 2 volume *Memoirs* (1855) is an expansion of this life.

Memoirs of the Life, Writing, and Discoveries of Sir Isaac Newton. By D. Brewster. 2 Volumes. Edinburgh, T. Constable and Company, 1855. Second Edition, Edinburgh, Edmonston and Company, 1860.

Sir Isaac Newton: A Brief Account of His Life and Work. By S. Brodetsky. London, Methuen and Company, Ltd., 1927. 165 pp.

Life of Sir Isaac Newton. By H. (Lord) Brougham. London, Library of Useful Knowledge, 1829. 40 pp.

Isaac Newton. By H. Crew. New York, Scripta Mathematica, 1943. 227 pp.

Essays on the Life and Work of Newton. By A. De Morgan. Edited with notes and appendices by P. E. B. Jourdain. Chicago, The Open Court Publishing Company, 1914. 198 pp.

Newton: His Friend: and His Niece. By A. De Morgan. Edited by S. E. De Morgan and A. C. Ranyard. London, E. Stock, 1885. 161 pp.

Newton: The Man. By R. De Villamil. London, G. D. Knox, 1931. 111 pp.

The Elogium of Sir Isaac Newton. By B. le Bovier de Fontenelle. London, J. Tonson, 1728. 32 pp.

The Life of Sir Isaac Newton; with an Account of His Writings. By B. le Bovier de Fontenelle. London, J. Woodman and D. Lyon, 1728. 26 pp.

The Life of Sir Isaac Newton, Containing an Account of His Numerous Inventions and Discoveries; and a Brief Sketch of Astronomy Previous to His Time, Compiled from Authentic Documents. By G. Grant. Dublin, J. M'Glashan, 1849. 311 pp.

Isaac Newton, 1642–1727. A Memorial Volume Edited for the Mathematical Association. By W. G. Greenstreet. London, G. Bell and Sons, Ltd., 1927. 181 pp.

Isaac Newton; a Biography. 1642–1727. By L. T. More. London and New York, Charles Scribner's Sons, 1934. 675 pp.

Sir Isaac Newton, A Biographical Sketch. By V. E. Pullin. London, E. Benn, 1927, 80 pp.

Newton and the Origin of Colours. By M. Roberts and E. R. Thomas. London, G. Bell and Sons, Ltd., 1934. 133 pp.

Matter and Gravity in Newton's Physical Philosophy; a Study in the Natural Philosophy of Newton's Time. By A. J. Snow. London, Oxford University Press, Humphrey Milford, 1926. 256 pp.

Memoirs of Sir Isaac Newton's Life by William Stukeley, 1752, Being Some Account of His Family and Chiefly of the Junior Part of His Life. By W. Stukeley. Edited by A. H. White. London, Taylor and Francis, Ltd., 1936. 86 pp.

Isaac Newton, 1642–1727. By J. W. N. Sullivan. New York, The Macmillan Company, 1938. 275 pp.

Correspondence of Sir Isaac Newton and Professor Cotes, Including Letters of Other Eminent Men, Now First Published from the Originals in the Library of Trinity College, Cambridge; Together with an Appendix Containing Other Unpublished Letters and Papers by Newton; with Notes, Synoptical View of the Philospher's Life, and a Variety of Details Illustrative of His History, by J. Edleston. London, J. W. Parker, 1850. 316 pp. For reference to other correspondence of Newton see *A Bibliography of the Works of Isaac Newton; Together with a List of Books Illustrating His Works, with Notes.* By G. J. Gray. Second Edition. Cambridge, Bowes and Bowes, 1907. 80 pp.

Sir Isaac Newton, 1727–1927, a Bicentenary Evaluation of His Work; a Series of Papers Prepared under the Auspices of the History of Science Society. . . . Edited by F. E. Brasch. Baltimore, The Williams and Wilkins Company, 1928. 351 pp.

William Oughtred, a Great Seventeenth-Century Teacher of Mathematics. By F. Cajori. Chicago, Open Court Publishing Company, 1916. 100 pp.

Pascal, the Life of Genius. By M. Bishop. New York, Reynal and Hitchcock, Inc., 1936. 398 pp.

Blaise Pascal. By J. Chevalier. Translated by L. A. Clare. London, Sheed and Ward; London and New York, Longmans, Green and Company, 1930. 336 pp. Essentially a study of Pascal's thought.

Pascal and the Port Royalists. By W. R. Clark. New York, C. Scribner's Sons, 1902, 235 pp.

The Physics of Pascal. By I. Leavenworth. New York, Publication of the Institute of French Studies, Inc., 1930. 164 pp.

Pascal; the Man and the Message. By R. H. Soltan. London, Blackie and Sons, Ltd., 1927, 216 pp.

Karl Pearson; an Appreciation of Some Aspects of His Life and Work. By E. S. Pearson. Cambridge, University Press, 1938. 170 pp.

Speeches Delivered at a Dinner Held in University College, London, in Honour of Professor Karl Pearson, 23 April 1934. Cambridge, privately printed, University Press, 1934.

Benjamin Pierce, 1809–1880: Biographical Sketch and Bibliography. By R. C. Archibald. Oberlin, The Mathematical Association of America; Chicago, The Open Court Publishing Company, 1925. 30 pp.

Benjamin Pierce . . . A Memorial Collection. Edited by M. King. Cambridge, Mass., 1881. 64 pp.

The Life-Romance of an Algebraist. By G. W. Pierce. Boston, J. G. Cupples, 1891. 167 pp. The writings of an eccentric.

Thomas Simpson and His Times. By F. M. Clarke. Baltimore and New York, Waverly Press, Inc., 1929. 215 pp.

Account of the Life and Writings of Robert Simson, M.D. By W. Trail. London, G. and W. Nicol, 1812. 190 pp.

James Stirling: a Sketch of His Life Along with His Scientific Correspondence. By C. Tweedie. Oxford, Clarendon Press, 1922. 213 pp.

The Mathematical Work of John Wallis, D.D., F.R.S. 1616–1703. By J. F. Scott. London, Taylor and Francis, Ltd., 1938. 240 pp.

The Life and Times of James Wilkins, Warden of Wadham College, Oxford. By P. A. Wright-Henderson. Edinburgh and London, W. Blackwood and Sons, 1910. 131 pp.

COLLECTED BIOGRAPHIES

Men of Mathematics. By E. T. Bell. New York, Simon and Schuster, 1937. 592 pp. Zeno, Archimedes, Pythagoras, Descartes, Fermat, Pascal, Newton, Leibniz, The Bernoullis, Euler, Lagrange, Laplace, Monge, Fourier, Poncelet, Gauss, Cauchy, Lobatchewsky, Abel, Jacobi, Hamilton, Galois, Sylvester, Cayley, Wierstrass, Kowalewski, Boole, Hermite, Kronecker, Riemann, Kummer, Poincaré, Cantor.

Memoirs of the Royal Academy of Sciences in Paris Epitomized, with the Lives of the Late Members of That Society and a Preface by Monsieur Fontenelle, Secretary and Author of the History of Said Academy. By B. le Bovier de Fontenelle. Translated by J. Chamberlayne. London, W. and J. Innys, 1721. 464 pp. Viviani, L'Hospital, James Bernoulli, Duhamel, Tschirnhaus, Carrée, Cassini among others.* There is a 1717 edition with a longer, somewhat different title.

* The phrase, "among others," signifies that these latter are not mathematicians.

Portraits of Famous Philosophers Who Were Also Mathematicians, with Biographical Accounts. By C. J. Keyser. New York, Scripta Mathematica, 1939. 52 pp. Pythagoras, Plato, Aristotle, Epicurus, R. Bacon, Descartes, Pascal, Spinoza, Leibniz, Berkeley, Kant, C. Peirce.

Lectures on Ten British Mathematicians of the Nineteenth Century. By A. Macfarlane. New York, John Wiley & Sons, Inc., 1916. 148 pp. Peacock, De Morgan, Hamilton, Boole, Cayley, Clifford, H. J. S. Smith, Sylvester, Kirkman, Todhunter.

Biographia Philosophica. Being an Account of the Lives, Writings, and Inventions of the Most Eminent Philosophers and Mathematicians Who Have Flourished from the Earliest Ages of the World to the Present Time. By B. Martin. London, W. Owen and B. Martin, 1764. 565 pp. Biographical sketches of some 150 men, including Greek and early English mathematicians not mentioned in more modern works.

Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Works. By G. Prasad. 2 Volumes. Benares, The Benares Mathematical Society, 1933. Vol. 1. Gauss, Cauchy, Abel, Jacobi, Weierstrass, Riemann. Vol. 2. Cayley, Hermite, Kronecker, Brioschi, Cremona, Darboux, Cantor, Mittag-Leffler, Klein, Poincaré.

Correspondence of Scientific Men of the Seventeenth Century, Including Letters of Barrow, Flamsteed, Wallis, and Newton. Printed from the Originals in the Collection of the Right Honourable the Earl of Macclesfield. By S. P. Rigaud. 2 Volumes. Oxford, University Press, 1841. See also, *Contents of the Correspondence of Scientific Men of the Seventeenth Century.* Compiled by A. De Morgan. Oxford, University Press, 1862. 36 pp.

A Portfolio of Portraits of Eminent Mathematicians. Edited by D. E. Smith. Chicago, The Open Court Publishing Company, 1905. Descartes, Pythagoras, Archimedes, Fermat, Leonardo of Pisa, Euclid, Cardan, Leibnitz, Napier, Vieta, Newton, Thales.

Portraits of Eminent Mathematicians; with Brief Biographical Sketches. By D. E. Smith. New York, Scripta Mathematica, 1936. Archimedes, Copernicus, Vieta, Galileo, Napier, Descartes, Newton, Leibniz, Lagrange, Gauss, Lobachevsky, Sylvester.

Portraits of Eminent Mathematicians with Brief Biographical Sketches. Portfolio No. 2. By D. E. Smith. New York, Scripta Mathematica, 1938. Euclid, Cardan, Kepler, Fermat, Pascal, Euler, Laplace, Cauchy, Jacobi, Hamilton, Cayley, Chebishef, Poincaré.

The Great Mathematicians. By H. W. Turnbull. London, Methuen and Company, 1929. 128 pp. The Greeks, Ferro, Fontana, Cardan, Vieta, Napier, Descartes, Fermat, Desargues, Pascal, Brouncker, Wallis, Gregory, Newton, The Bernoullis, Euler, Maclaurin, Lagrange, Gauss, Hamilton.

Biographies of Men of Science. By T. Young. Volume 2 of his collected *Miscellaneous Works*, see below. Lagrange, Fermat, Lambert among others.

COLLECTED WORKS

Pure Mathematics

Collected Scientific Papers of John Aitken. Edited by C. G. Knott. Cambridge, University Press, 1923. 591 pp. Contains a memoir by C. G. Knott.

The Collected Logical Works of George Boole. 2 Volumes. Vol. 1 unpublished. Vol. 2. *The Laws of Thought.* London and Chicago, The Open Court Publishing Company, 1916; photo-reprint, 1940.

The Collected Mathematical Papers of Arthur Cayley. 14 Volumes. Cambridge, University Press, 1889–1899. Vol. 8 contains a memoir by A. R. Forsyth.

Lectures and Essays by the Late William Kingdon Clifford, F.R.S. Edited by L. Stephen and F. Pollock. London, The Macmillan Company. 2 Vols., 1879; 1 Vol., 1886; 2 Vols., 1901. Vol. 1 contains a memoir by F. Pollock; those in the 1886 and 1901 editions are identical and slightly longer than that in the 1879 edition.

The Mathematical Papers of William Kingdon Clifford. Edited by R. Tucker. London, Macmillan and Company, 1882, 658 pp. Contains memoirs by R. Tucker and H. J. S. Smith.

Mathematical Researches of Sir James Cockle, M.A., F.R.S., Chief Justice of Queensland, 1864–77. Reprints from various journals; the copy available to me had no data *re* publisher, year, *etc.*, nor have I been able to ascertain such.

The Mathematical and Other Writings of Robert Leslie Ellis. Edited by W. Walton. Cambridge, Deighton, Bell and Company; London, Bell and Daldy, 1863. 427 pp. Contains a memoir by H. Goodwin.

The Mathematical Writings of Duncan Farquharson Gregory. Edited by W. Walton. Cambridge, Deighton, Bell and Company, 1865. 267 pp. Contains a memoir by R. L. Ellis.

The Mathematical Papers of Sir William Rowan Hamilton. Vol. 1 edited by A. W. Conway and J. L. Synge; Vol. 2 edited by A. W. Conway and A. J. McConnell. Cambridge, University Press, 1931, 1940. Vol. 1 contains a memoir by C. Graves.

The Collected Works of George Abram Miller. Urbana, University of Illinois Press, 1935– . Vols. 1 and 2 have been issued to date.

The Mathematical and Physical Papers of B. O. Peirce, 1903–1913. Cambridge, Harvard University Press, 1926. 444 pp.

Collected Papers of Charles Sanders Peirce. 6 Volumes. Edited by C. Hartshorne and P. Weiss. Cambridge, Harvard University Press, 1931–1935.

Collected Papers of Srinivasa Ramanujan. Edited by G. H. Hardy, P. V. S. Aiyer and B. M. Wilson. Cambridge, University Press, 1927. 355 pp. Contains memoirs by R. V. Sesha Aiyar and R. B. Rao and by G. H. Hardy. See also *Ramanujan.* By G. H. Hardy, Cambridge, University Press, 1940. 236 pp. This text contains valuable comment on Ramanujan's work.

The Collected Mathematical Papers of Henry John Stephens Smith. Edited by

J. W. L. Glaisher. 2 Volumes. Oxford, Clarendon Press, 1894. Contains memoirs by C. H. Pearson, B. Jowett, Lord Bowen, J. L. Strachan-Davidson, and A. Robinson.

The Collected Mathematical Papers of James Joseph Sylvester. Edited by H. F. Baker. 4 Volumes. Cambridge, University Press, 1904–1912. Vol. 4 contains a memoir by H. F. Baker.

Students' [William Sealy Gosset] *Collected Papers.* Edited by E. S. Pearson and J. Wishart. Cambridge, University Press, 1942. 244 pp. Contains a memoir by L. McMullen.

Mathematical Astronomy

The Scientific Papers of John Couch Adams. Vol. 1 edited by W. G. Adams. Vol. 2, part 1 edited by R. A. Sampson, part 2 edited by W. G. Adams. Cambridge, University Press, 1896, 1900. Vol. 1 contains a memoir by J. W. L. Glaisher.

Scientific Papers of Sir George Howard Darwin. Edited by F. J. M. Stratton and J. Jackson, 5 Volumes. Cambridge, University Press, 1907–1916. Vol. 5 contains memoirs by F. Darwin and E. W. Brown.

The Scientific Papers of Sir William Herschel. Including Early Papers Hitherto Unpublished with a Biographical Introduction Compiled Chiefly from Unpublished Material by J. L. E. Dreyer. 2 Volumes. London, The Royal Society and the Royal Astronomical Society, 1912.

The Collected Mathematical Works of George William Hill. 4 Volumes. Washington, Carnegie Institution of Washington, 1905–1907. Vol. 1 contains a memoir in French by H. Poincaré.

Scientific Papers of William Parsons, Third Earl of Rosse, 1800–1867. Republished by Sir C. Parsons. Bradford and London, P. Lund, Humphries and Company, Ltd., 1926. 221 pp.

Mathematical Physics

The Scientific Papers of the Late Thomas Andrews. London, Macmillan and Company, 1889. 514 pp. Contains memoirs by P. G. Tait and A. C. Brown.

The Works of Archimedes. Edited in modern notation by T. L. Heath. Cambridge, University Press, 1897. 326 pp. Chapter I of the "Introduction" comprises a memoir by T. L. Heath.

The Method of Archimedes Recently Discovered by Heiberg; a Supplement to the Works of Archimedes, 1897. Edited by T. L. Heath. Cambridge, University Press, 1912. 51 pp.

Collected Physical Papers of Sir Jagadis Chunder Bose. London, Longmans, Green and Co., 1927. 404 pp.

The Philosophical Works of the Hon. Robert Boyle Esq.; Abridged, Methodized, and Disposed under the General Heads of Physics, Statics, Pneumatics, Natural History, Chemistry and Medicine. Second edition, 3 Volumes. London, Peter Shaw, 1738.

The Works of the Honourable Robert Boyle. 5 Volumes. London, A. Millar, 1744. 6 Volumes. Vol. 1, London, J. F. Rivington; Vols. 2–6, London, W. Johnston, 1772. Vol. 1 of each edition contains "The Life of the Honourable Robert Boyle" by Thomas Birch.

The Collected Papers of George Ashley Campbell. Printed by Lancaster Press, Lancaster, Pa., for the American Telephone and Telegraph Company, New York, 1937. 586 pp. Contains memoirs by V. Bush and E. H. Colpitts.

The Electrical Researches of the Honourable Henry Cavendish, F.R.S., Written Between 1771 and 1781. Edited by J. C. Maxwell. Cambridge, University Press, 1879. 454 pp. Contains a memoir by J. C. Maxwell.

The Scientific Papers of the Hon. Henry Cavendish. Volume 1 edited by J. C. Maxwell, revised by J. Larmor; Volume 2 edited by T. E. Thorpe. Cambridge, University Press, 1921. Vol. 1 contains a memoir by J. C. Maxwell.

Collected Papers of Sir James Dewar. Edited by Lady Dewar and J. D. H. Dickson, H. M. Ross, and E. C. S. Dickson. 2 Volumes. Cambridge, University Press, 1927.

Scientific Memoirs, Being Experimental Contributions to a Knowledge of Radiant Energy. By J. W. Draper. London, Sampson Low, Marston, Searle and Rivington; Harper and Bros., 1878. 467 pp.

Experimental Researches in Chemistry and Physics. By M. Faraday. London, R. Taylor and W. Francis, 1859. 496 pp.

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TANGENT TRIANGLES TO A BIQUADRATIC CURVE

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1. Sketching a biquadratic curve whose inflection points are real. Ask a sophomore calculus student how to sketch the graph of a biquadratic polynomial, and he might answer "Set the derivative equal to zero, locate the maximum and minimum points, and then sketch the curve." One difficulty with this procedure is, that unless the biquadratic has been especially fixed up to have a derivative with integral roots—or at the worst, rational roots—the student may find himself confronted with an irreducible cubic equation which he does not

know how to solve. Furthermore, the maximum- and minimum-point procedure fails to show a certain symmetry of the biquadratic curve which will be noted below.

Restricting our discussion to *biquadratic polynomial graphs*, $y=f(x)$, *having real inflection points*, we shall show how certain important tangent lines with their contact points may be located geometrically after finding both coordinates of each of the inflection points $(h\pm a,\ k+4b\pm ma)$ and the ordinate $f(h)=k+9b$ midway between these—five independent quantities involving at most quadratic irrationalities in terms of the five coefficients of the polynomial. Six important tangents form a characteristic configuration of two intersecting triangles, similar in the ratio one to two, which guide the curve as it bends away from its inflection points. Two other tangents give the directions of the curve further out. Sketching in the curve from these eight tangents is then a simple matter.

2. Affine reduction to normal form. The general biquadratic polynomial graph with real inflection points has an equation of the form

$$y = f(x) = c_0x^4 + 4c_1x^3 + 6c_2x^2 + 4c_3x + c_4, \qquad c_1^2 > c_0c_2. \tag{1}$$

This can be transformed into the particular biquadratic polynomial

$$Y = F(X) = X^4 - 6X^2 + 9, \tag{2}$$

by first shifting to a suitable new origin (h, k), then subtracting a linear term $m(x-h)$, and then changing the scale on the new axes by factors a and b respectively. Five constants h, k, m, a, b are involved, which are expressible in terms of the five coefficients $c_0, c_1, \cdots, c_4$ as follows:

$$\begin{aligned} &a = \sqrt{c_1^2 - c_0c_2}/c_0, \qquad b = c_0a^4, \\ &h = -c_1/c_0, \qquad k = f(h) - 9b, \qquad m = f'(h). \end{aligned} \tag{3}$$

The explicit equations of transformation are

$$\begin{aligned} aX &= x - h, & x &= h + aX, \\ bY &= (y - k) - m(x - h), & y &= k + bY + maX, \\ \frac{dy}{dx} &= \frac{b}{a}\frac{dY}{dX} + m. \end{aligned} \tag{4}$$

Important properties of this transformation (4) which is called an *affine transformation preserving vertical lines*, are the preservation of collinearity of points, concurrency or parallelism of lines, midpoints, centroids, tangents and contact points, and ratios of distances along parallel lines. A certain configuration of tangents to the simple biquadratic curve (2) leads to a corresponding configuration for the general biquadratic curve (1) which makes it easy to sketch.

3. Theorems on centroid points. We start by proving a theorem defining an important centroid point C associated with a biquadratic curve (Fig. p. 449).

THEOREM 1. *The centroid C of the contact points, on the three tangents of a given slope* λ *which can be drawn to a given biquadratic curve, is a fixed point independent of* λ.

Proof. Consider first the biquadratic (2) and let the three contact points on tangents of slope λ be $(X_i Y_i)$, $i=1, 2, 3$.

Then X_1, X_2, X_3 are roots of the equation $dY/dX-\lambda=0$, namely

$$4X^3 - 12X - \lambda \equiv 4(X - X_1)(X - X_2)(X - X_3) = 0. \tag{5}$$

Comparing coefficients of X^2 and X, respectively, we have

$$\begin{gathered} X_1 + X_2 + X_3 = 0, \\ 2(X_1X_2 + X_1X_3 + X_2X_3) = -6 = (X_1 + X_2 + X_3)^2 - (X_1^2 + X_2^2 + X_3^2). \end{gathered} \tag{6}$$

Hence the coordinates of the centroid C are $(0, 3)$, for all λ, since

$$\begin{aligned} &\tfrac{1}{3}(X_1 + X_2 + X_3) = 0, \quad Y_i = (X_i^4 - 3X_i^2) - 3X_i^2 + 9 = \lambda X_i/4 - 3X_i^2 + 9, \\ &\tfrac{1}{3}(Y_1 + Y_2 + Y_3) = (\lambda/12)(X_1 + X_2 + X_3) - (X_1^2 + X_2^2 + X_3^2) + 9 \\ &\qquad = 0 - 6 + 9 = 3. \end{aligned} \tag{7}$$

Since tangency, parallelism, and centroids are unchanged by the transformation (4), the transform of the point C, whose coordinates are $(h, k+3b)$, will be the centroid of contact points for parallel tangents of biquadratic (1).

There are just three slopes for which two of the three parallel tangents coincide. The doubly counting tangents are the two inflectional tangents d_1 and d_2, each having a doubly counting contact at one the inflection points A_1 and A_2; and double tangent d, having two distinct contact points B_1 and B_2. (We call the segment B_1B_2 the *base* and denote its midpoint by B.) The three doubly counting tangents form a triangle $D\ D_1D_2$. Parallel to each of the sides d_i of this triangle there is but one other (singly counting) tangent s_i, and the three tangents s, s_1, s_2 form a triangle $S\ S_1\ S_2$ similar to triangle $D\ D_1D_2$ but twice as large. The contact points M, C_1, C_2 are determined from the centroid theorem proved above. Since A_1 and A_2 are doubly counting as contact points, and B is midway between B_1 and B_2, the segments BM, A_1C_1, A_2C_2 are trisected by C.

The vertical line through the base center B we call the *axis* of the biquadratic curve, and the point M where it cuts the curve we call the *midpoint* of the curve. The axis passes through the centroid C and through the inflectional center A, midway between the inflection points. It also contains the vertices D and S of the two principal tangent triangles. An important property of the axis is based on the following theorem.

THEOREM II. *The centroid of the four points (real or imaginary) in which an arbitrary line cuts a biquadratic polynomial curve is a point on a vertical line called the axis, which is defined by setting the third derivative of the biquadratic polynomial equal to zero.*

Proof. The intersections x_1, x_2, x_3, x_4 of an arbitrary line $y=px+q$ with the biquadratic curve (1) are given by the equation

$$c_0x^4 + 4c_1x^3 + 6c_2x^2 + 4c_3x + c_4 - px - q = c_0(x - x_1)(x - x_2)(x - x_3)(x - x_4) = 0. \tag{8}$$

Comparing coefficients of x^3, we have $4c_1=c_0(-x_1-x_2-x_3-x_4)$. Hence the average of the x_i is given by

$$x = \tfrac{1}{4}(x_1 + x_2 + x_3 + x_4) = -c_1/c_0 = h, \tag{9}$$

which is easily seen to be the value of x for which the third derivative $24c_0x + 24c_1$ is zero.

Let the arbitrary line be in particular one of the inflectional tangents d_i which intersects the curve three times at the inflection point A_i, and cuts the axis at D. Then its fourth intersection G_i is a point on d_i such that $\overline{DG_i}=3\overline{A_iD}$. The two points G_1 and G_2 so determined are useful in sketching the curve beyond the contact points C_1, C_2 already determined.

4. Coordinates of special points. Returning to the special biquadratic (2) we next compute by elementary calculus and analytic geometry the coordinates of the various points and the slopes of the various tangents we have discussed. We have

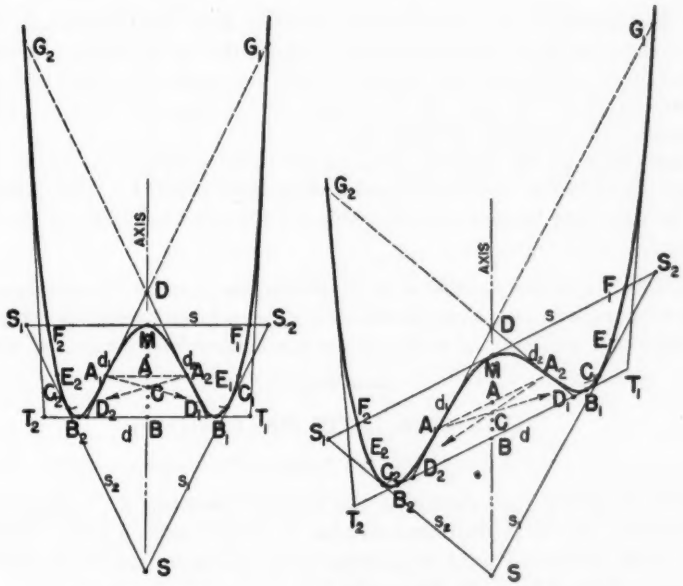
$$Y' = 4X^3 - 12X, \qquad Y'' = 12X^2 - 12. \tag{10}$$

The inflection points A_1, A_2 are at $(\mp 1, 4)$ and the slopes of d_1 and d_2 are ± 8. Hence the inflectional center A is at $(0, 4)$, and the vertex D is at $(0, 12)$. The double tangent d is the x-axis, having slope 0, its contact points B_1, B_2 are at $(\pm\sqrt{3}, 0)$, and the vertices D_1, D_2 of the doubly tangent triangle are at $(\pm 1.5, 0)$. The midpoint M is at $(0, 9)$, the base center B is at $(0, 0)$ and the centroid C is at $(0, 3)$, (as we proved). Since C trisects the segments $\overline{A_iC_i}$, the points C_1, C_2 are at $(\pm 2, 1)$. Since the slopes at these contact points are the inflectional slopes ± 8, the point S is at $(0, -15)$, while S_1, S_2 are at $(\mp 3, 9)$. The line A_1A_2 through the inflection points is parallel to the double tangent and intersects the biquadratic again in points E_1, E_2 at $(\pm\sqrt{5}, 4)$. The tangent line at the midpoint M is parallel to A_1A_2 and meets the biquadratic again in points F_1, F_2 at $(\pm\sqrt{6}, 9)$. The inflectional tangents intersect the biquadratic again in points G_1, G_2, at $(\pm 3, 36)$ respectively, and since the corresponding slopes are ± 72, respectively, the tangents g_1, g_2 intersect the base line at points T_1, T_2 with coordinates $(\pm 2.5, 0)$. Points and slopes are given in the following table:

Auxiliary points:	A	B	C	D	D_1	D_2	S	S_1	S_2	T_1	T_2
X-coordinates:	0	0	0	0	1.5	−1.5	0	−3	3	2.5	−2.5
Y-coordinates:	4	0	3	12	0	0	−15	9	9	0	0

Points on Curve:	G_2	F_2	E_2	C_2	B_2	A_1	M	A_2	B_1	C_1	E_1	F_1	G_1
X-coordinates:	−3	$-\sqrt{6}$	$-\sqrt{5}$	−2	$-\sqrt{3}$	−1	0	1	$\sqrt{3}$	2	$\sqrt{5}$	$\sqrt{6}$	3
Y-coordinates:	36	9	4	1	0	4	9	4	0	1	4	9	36
Slope:	−72			−8	0	8	0	−8	0	8			72

5. Geometric construction of the tangent configuration. The corresponding points and slopes on the arbitrary biquadratic curve (1) are obtainable from these by means of the transformation (4), but it is easier to sketch the configuration without using the explicit coordinates of any points except the inflection points A_1, A_2 and the midpoint M, which are computed from the original function. Having located the points A_1, A_2, M, we proceed geometrically to draw the configuration as follows:



$Y=X^4-6X^2+9$. The biquadratic polynomial graph.

Step. 1. Locate A, midway between A_1 and A_2.

Step. 2. Locate C on the vertical axis MA extended through A so that $\overline{AC}=\frac{1}{2}\overline{MA}$.

Step 3. Locate B beyond C on MA so that $\overline{AB}=4\overline{AC}$.

Step 4. Locate D on AM extended through M so that $\overline{DM}=\overline{CB}$. (Check: $2\overline{DM}=\overline{MC}$.)

Step 5. Locate S on the axis MA beyond B so that $\overline{CS}=2\overline{DC}=2\overline{MB}$.

Step 6. Locate C_1 and C_2 by extending A_1C and A_2C each twice its own length through C.

Step 7. Draw tangents A_1D, A_2D, and parallel tangents C_1S and C_2S. (Check parallelism.)

Step 8. Complete the triangles DD_1D_2 and SS_1S_2 by drawing parallels to A_1A_2 through B and M. (Check: $\overline{S_1M}=\overline{MS_2}=\overline{D_2D_1}$.)

Step 9. Locate G_1 and G_2 by extending A_1D and A_2D each three times its length through D.

Step 10. Locate T_2 and T_1 on D_2D_1 (but not between D_2 and D_1), so that $\overline{T_2D_2}=\overline{D_1T_1}=\overline{A_1A}$; and draw the tangents T_1G_1 and T_2G_2.

Step 11. Estimate contact points B_2, B_1 on the base D_2D_1 at distances $\sqrt{3}\ \overline{A_1A}$ from the axis; and also the intersections E_2, E_1 at distances $\sqrt{5}\ \overline{A_1A}$ from axis on A_1A_2, and the intersections F_2, F_1 at distances $\sqrt{6}\ \overline{A_1A}$ from axis on S_1S_2.

Step 12. Sketch in the curve from A_1 to M to A_2 inside the triangle DD_1D_2, using tangents at these three points and making the curve concave towards A. Then use the other constructed contact points and tangents on the two arms of the curve and draw the curve to be concave in the opposite direction (upward if b is positive, downward if b is negative).

A final check on the drawing is based on a certain axial symmetry of a biquadratic curve when viewed obliquely along lines parallel to the double tangent. The following theorem defining this "symmetry" is left as an exercise to the reader.

THEOREM III. *A line parallel to the double tangent which intersects a biquadratic polynomial curve in real points, intersects it either in one pair or in two pairs of points, such that the points of each pair are equidistant from the axis of the curve.*

THREE-LINE LATIN RECTANGLES

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This note shows the relation of the reduced *problème des ménages* to the enumeration of three-line Latin rectangles. A simple solution of the former has recently been published by I. Kaplansky [1]; the latter has been studied by S. M. Jacob [2] and by S. M. Kerawala [3].

A Latin rectangle is an array in which each row is a permutation of elements 1 to n and each column has distinct elements.

The *problème des ménages* asks for the number of ways of seating n married couples, husbands alternating with wives, at a circular table, so that no husband sits next his wife. Fixing positions of wives (or husbands) the problem is reduced to the enumeration of permutations discordant with the two permutations:

$$\begin{array}{cccccc} 1 & 2 & 3 \cdots & n-1 & n \\ 2 & 3 & 4 \cdots & n & 1 \end{array}$$

or, more generally, to the enumeration of permutations discordant with two permutations, one of which is the identity, the other a cycle of n.

This enumeration may be used for three-line Latin rectangles in the following way. With the identity as the first line (which introduces an $n!$ factor in the enumeration) the second line may be any permutation discordant with the identity. The number of these is known to be the subfactorial of n, or $\Delta^n 0!$,

where Δ is the finite difference operator. Then for each of these the number of permutations discordant with both lines is enumerated as a linear function of *problème des ménages* numbers.

Consider in cycle form, the second line permutations, since they leave no element unchanged, contain no cycles of degree one, that is, the cycle structure is a partition of n without unit parts; *e.g.*, for $n=4$, the permutations and their cycle structure are as follows:

Perm.	*Cycle Struct.*	*Perm.*	*Cycle Struct.*	*Perm.*	*Cycle Struct.*
2143	2^2	3142	4	4123	4
2341	4	3412	2^2	4312	4
2413	4	3421	4	4321	2^2

By a known formula [4], the number of permutations of cycle structure $2^{a_2}3^{a_3}\cdots n^{a_n}$, where $(a_2, a_3, \cdots, a_n)$ is some solution of $2a_2+3a_3+\cdots+na_n=n$, is:

$$C_a = \frac{n!}{2^{a_2}a_2!\cdots n^{a_n}a_n!}. \tag{1}$$

Kaplansky's solution of the *problème des ménages* may be written:*

$$u_n = \sum_{i=0}^{n} (-1)^i \frac{2n}{2n-i}\binom{2n-i}{i} E^{n-i}0!, \tag{2}$$

where E is the shift operator of finite differences: $Ef_n=f_{n+1}$, $E0!=1!$ $E^{n-i}0!=(n-i)!$. Thus $u_n \equiv U_n(E)0!$, with $U_n(E)$ a polynomial of degree n in E.

The number of permutations discordant with both the identity and a permutation of class $a=(a_2, a_3, \cdots, a_n)$ is (*cf.* Kaplansky [5]):

$$u_{[a]} = U_2^{a_2}U_3^{a_3}\cdots U_n^{a_n}0!. \tag{3}$$

In this symbolical expression, E is treated as an ordinary algebraical quantity until all operations are completed.

Then the number of three-line Latin rectangles is

$$_3L_n = n!{}_3K_n = n!\sum C_a u_{[a]} = \sum \frac{n!^2 U_2^{a_2}\cdots U_n^{a_n}0!}{2^{a_2}a_2!\cdots n^{a_n}a_n!}, \tag{4}$$

where the summation is over all partitions of n without unit parts.

Now U_n has the recurrence:

$$U_n = (E-2)U_{n-1} - U_{n-2}, \tag{5}$$

which, with the substitution $e=E-2$, is the recurrence for Tchebycheff polynomials, and in fact U_n may be written:

* A more compact form is $u_n=2\cos(2ny)$ with $2\cos y=\sqrt{v}$ and $v^n=E^n0!=n!$, as given without proof by Touchard [6]; this is an immediate consequence of equation (6) below.

$$U_n = 2\cos(nx), \qquad \cos x = \tfrac{1}{2}e. \tag{6}$$

This agrees with (2) except for the initial values U_0 and U_1, where the function has no meaning in the enumeration sense. For the use of (6) which follows, these initial values are $U_0=2$, $U_1=e=E-2$.

Then,

$$\begin{aligned} U_iU_j0! &= [4\cos ix\cos jx]0! \\ &= u_{i+j} + u_{i-j}. \end{aligned} \tag{7}$$

which is readily generalized to

$$U_{i_1}U_{i_2}\cdots U_{i_k}0! = \sum u_{i_1\pm i_2\pm\cdots i_k} \tag{8}$$

(*cf.* Touchard [6]), where the sum is over the 2^{k-1} possible assignments of $+$ and $-$ signs.

Hence (3) may be evaluated as a sum of values of u with proper subscripts. (Note that for interpretation of (7) or (8), $u_{-n}\equiv u_n$.)

This evaluation may be used in (4) without difficulty for the smaller values of n; *e.g.*, for $n=4$, the results may be tabulated as follows:

a	C_a	$u_{[a]}$	$C_au_{[a]}$
4	6	2	12
2^2	3	4	12

so that ${}_3K_4=24$, in agreement with Jacob and Kerawala.

Also the behavior for large values of n follows immediately on noting that, for all possible $a=(a_2, a_3, \cdots, a_n)$

$$u_{[a]} \sim u_n \sim n!e^{-2}.$$

Then ${}_3K_n$ behaves like the product of u_n and the subfactorial of n, which for n large is $n!e^{-1}$; that is,

$${}_3K_n \sim (n!)^2e^{-3}, \tag{9}$$

a result which Kerawala surmised but failed to prove, though his numerical evidence was practically conclusive (agreeing with e^{-3} to seven decimal places).

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MATHEMATICS FOR STUDENTS OF THE HUMANITIES

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1. Mathematics and the liberal arts. Most colleges are looking forward already to the postwar period; new plans are being discussed and the old curricula are coming under close scrutiny. Undoubtedly the teaching of mathematics will be fundamentally affected in this process of revaluation and reorganization. In the college faculties we shall experience searching discussions in regard to the role which mathematics should naturally play in a liberal arts curriculum.

Historically mathematics always constituted an integral part of the liberal arts, in all the meanings which this term has had through the centuries. There is every indication that it will retain this position in the future. In the present scientific trend of our civilization and education the mathematical theories represent not only a body of important facts, but they constitute also a language without which many phases of scientific thought would be cumbersome to represent and difficult to understand. Furthermore, mathematics embodies the framework for the theory of logic and other philosophic constructions. Mathematical terminology and methods of expression gradually are becoming more prevalent. Even the daily press and publications which usually strive to present their subject matter in an entirely popular form do not shrink from using mathematical terms. For this reason there does not seem to be any trend away from the inclusion of mathematics in the liberal education; perhaps one may expect a gradually strengthened position. This situation makes it imperative that the position of mathematics in the educational scheme should be closely analyzed so that a well considered basis for its role in a liberal arts curriculum can be established.

2. Relation of students to mathematics. The students in a college usually fall into different groups in regard to their interests and their studies of mathematics.

One important group consists of those students who are to become engineers or intend to specialize in various physical sciences or to major in mathematics. Most of these men are aware of the fact that their studies will require mathematics of a rather advanced character and also that proficiency in mathematical manipulations and knowledge of technical procedures will be of great value to them. Most of the students in this group will take several courses in mathematics and some will extend their study of the subject throughout their whole period of attendance. Many of the courses given for these men are well defined through the prospective applications, and these students usually constitute the great majority in the advanced mathematics courses in the curriculum. There are, of course, at most institutions improvements which could be made in the instruction for these men. Certainly their courses must be revised from time to time according to needs and applications and according to the development of

new mathematical ideas and points of view. To me it seems desirable, for ininstance, if possible, to broaden the basis for these courses with more emphasis on some of the fundamental mathematical ideas. This would probably be advantageous both from the pedagogical and scientific points of view, and it would certainly reduce some of the now so common repetitions in the advanced calculus courses where the previous theories must be put on a more satisfactory logical basis. However, the possibility of more emphasis on fundamental ideas is counteracted somewhat from the beginning by the fact that the instruction of engineers and science students must to a certain extent be subordinate to the requirements of some of the fields of application. Thus it may be necessary to bring the students very quickly to a certain level of proficiency in handling the differential and integral calculus and the instruction often becomes one-sidedly and narrowly directed at attaining this goal at the earliest possible moment.

In spite of these remarks it appears that from the point of view of instruction this group of mathematics students does not represent as difficult a problem as other large groups which we shall discuss presently. Science and engineering students will usually broaden their mathematical views through their several courses. Furthermore, the students in this class have an easier task both through the clearer understanding of the necessity of studying the subject and through their natural interest and gifts for mathematics which enable them to supplement the instruction by independent studies. Therefore, I wish to make it quite clear, that the following considerations are not essentially concerned with this group of students.

In any liberal arts college there will be large groups of students whose relations to the field of mathematics are entirely different trom those of the science students and engineers. To illustrate this in specific figures, some statistics from Yale College extending over some years before the war may be quoted: Close to 60 per cent of the students take no mathematics course whatever, about 35 per cent take one course only, while a mere 5 per cent take more than one course. (In order that these figures shall not be misleading it may be stated that Yale College includes very few science students, since these men usually graduate from the Sheffield Scientific School.)

If one analyzes the group of students who take one course in mathematics, one finds that many take another science course—physics, chemistry, biology, geology, psychology, *etc.* Some intend to study medicine or major in economics. In all these fields a certain familiarity with the mathematical procedures and methods is useful so that a basic course in mathematics often is made a requirement. It may be recalled specifically, that a particular method which is common to many of these fields is statistical analysis, and its increased use makes it extremely desirable that its terminology and basic principles be known to the students. To this group of students taking one course in mathematics should be added a considerable number of students specializing in the humanities, who correctly feel that a liberal education is not complete without some knowledge of mathematics. Some of them like the subject well enough, but there are many

among them, who have but little gift for mathematics, and nevertheless are willing to go through its supposed ordeals for the cultural value and the general usefulness of the subject.

This brings us finally to the great number of students who prefer to stand on the sidelines as far as mathematics is concerned. In attempting to discover their reasons, one finds that this is not always because of an unwillingness to take mathematics. Many of them have a real desire to gain some mathematical knowledge and feel that it would be advantageous in their education, but they fear that they are not capable of fulfilling the requirements of the courses, so that they avoid difficulties and unhappiness by turning to other subjects.

3. Organization of a special course. When it is recognized that this situation exists one must ask the question: On which principles should a single mathematical college course be organized so that it serves as well as possible the educational interests of those students for whom it will most likely represent the end of their mathematical studies? At present many institutions make no particular provision for these men. They are simply offered one or two courses which are identical with those given to beginning science and engineering students. For many students this may be too difficult, but a much more important deficiency lies in the fact that the students after such a course are left with an impression confirming their previous school experiences that mathematics is mainly a technical tool which requires some special mental skill. They are presented with no broader view of mathematics and given little understanding of mathematics as a method and as an important branch of human thought and philosophy developed through centuries.

There are certain principles, it seems to me, which offer themselves naturally as guides in the organization of a terminal one-year course in mathematics.

First, as a general remark it is desirable to give the course a larger measure of broadness with greater emphasis on first principles than those courses now provide.

Secondly, as every teacher of mathematics knows, the ability to solve fairly complicated classroom problems is very short-lived. Most students after graduation retain only a vague memory of mathematics and little interest in the subject remains. In the choice of subject matter one should therefore attempt to include some information on problems which they may encounter later in everyday life and conversation.

A third basic aim of the course should be to present the development of mathematics as an important branch in the history of thought. For this purpose it is essential that the historical side of the subject should be touched upon in connection with all topics which are discussed. This, in my experience, contributes in no small measure to the interest of the students, and it serves the additional purpose of breaking the dry presentation of the average mathematics course.

Some of the classical problems which occasionally appear within a cultured

man's sphere of interest should not be omitted. For instance, a college graduate should not use the term "squaring of the circle" without having an idea of what this problem involves. Similarly, the teacher should at times step out of the beaten path of the subject to attempt to give the students a view of those peaks of scientific achievement which have already been reached, or some of the unsolved problems on which the interests of mathematicians have been focussed.

4. Content of the course. After these general considerations let us turn to more definite proposals for the content of a course of this nature. It would seem desirable to begin with a discussion of the basic concept, the number system from integers to real numbers and complex numbers. There is no reason why imaginary numbers should be unintelligible to a college student. The Greek dilemma in regard to the real numbers serves as an illustration of the decisive importance which a single idea may have. Trigonometry, with its welter of special formulas which even the teacher often has difficulty in remembering, should be reduced to a bare minimum, probably not more than the simplest facts for a right triangle with an indication of how other figures can be handled. A brief review of analytic geometry would probably be necessary with only one or two of the simplest curves discussed. A rapid excursion into space provides an opportunity to explain what the mathematician understands by higher dimensions so that the concept of fourth dimension can be stripped of the veil of mysticism with which it is usually draped in ordinary conversation.

Calculus must of course be introduced and its importance should be stressed. However, only the very simplest parts should be included in the actual course work. Differentiation and integration of simple polynomials are quite sufficient to illustrate the main ideas. Higher functions like exponentials, logarithms and trigonometric functions should not be touched upon. This is also a more straightforward acknowledgment of certain barriers than those pseudo-explanations which are now often used in this connection, stating conveniently at difficult points that "it can be shown." Simple maxima and minima problems should be treated since they usually impress and interest the students.

This brings us to some of the further topics which I should like to see included in the course and for which certainly some time will be available. Number theory is a subject of peculiar interest to many students. The Euclid algorithm is presumably known; it yields quite simply the solution of linear Diophantine equations and the unique factorization in primes. The infinity of primes, the distribution theorem for primes as well as the largest known primes, should certainly be mentioned in this connection. It adds to the interest of the students to point out that a great number of the puzzles which commonly appear in magazines can be solved by linear Diophantine equations. The sum of the divisors of a number is easily obtained, and it enables one to explain the numerological mysticisms in Plato in regard to perfect numbers and amicable numbers. Practically no students are acquainted with simple rules for checking multiplications and divisions by casting out nines and other rules. Perhaps this may all

appear as a somewhat long tribute to number theory, yet all the topics mentioned here can be presented in a few classroom hours.

An excursion into the theory of equations and its long history is quite fascinating. The only drawback is that the classroom work would be limited; only the solution of quadratic and cubic equations could be achieved. But one should explain that every equation has an many roots as its degree when the imaginary ones are counted. The historical milestones in the theory should be reported up to the result that not all equations can be solved by radical expressions. One should explain the fact that geometric constructions with compass and ruler lead to expressions involving square roots and thus one has gained an introduction to a brief discussion of the classical Greek problems of squaring the circle, trisecting the angle and doubling the cube. If one had gone so far in number theory as to determine the number of integers less than and relatively prime to a given one, even the results on the construction of regular polygons can be stated completely.

Another topic which is very inviting is the theory of probability. Here one has a field whose problems may occur in any one's experience and whose basic rules can be explained briefly and fully without the use of complicated mathematical apparatus. Some properties of combinations and permutations must be derived but this again seems advantageous since it leads to the kind of reasoning everyone runs into occasionally, be it in arranging dinner-guests or other objects. A variety of probability problems of interest to the students can be solved and the fact that they occur in connection with card games and gambling is in line with the history of the subject and should not cause any moral scruples.

There may be many other subjects which could be adapted to the course: those mentioned here should only be taken as specific examples; perhaps it may not be possible to find time even for all these. The detailed subject matter should depend to a considerable extent on the wishes and tastes of the instructor since the success of the course will depend largely upon his personal qualifications and enthusiasm. It requires a teacher who himself has broad interests in mathematics and an ability to transfer some of his own enthusiasm to the students. On the other hand, this freedom represents a challenge to the younger instructors which they, in my experience, will heartily welcome.

5. Concluding remarks. Of course there may be objections to the course as outlined, and it will probably be scorned as being "easy" by many of the old mathematical drill masters. If it is an easy course because it is entertaining and because a certain effort has been made to find topics which many of the students in this group can more readily absorb, then the charge is true. Too little effort is often being made in college courses to retain any of the quality of entertainment of mathematics which it undoubtedly had in many ancient cultures, particularly emphasized in Indian mathematics. The gentleman mathematicians of the seventeenth and eighteenth centuries who studied mathematical problems *con amore* in their spare time hardly exist any more and I feel this is to be de-

plored. However, if one means by an easy course, a course in which the students do little serious work, I think the charge will be without foundation. There are as many opportunities for serious problems and logical deliberation in the fields outlined above as in any others, and it should be insisted that those questions which are actually discussed from more than the historical side should actually be mastered. There is no intention that the course should be effortless on the part of the students.

And so a final remark. College instruction in this country, and for that matter in most other countries of the world, is faced with the problems of the scientific versus the humanistic education. Fortunately one must say, the question is not to be decided one way or the other. The present so common comparisons of values in the two fields, presumably to vindicate one field over the other, contributes nothing to the problem. Only a serious attempt to bring the two in contact will bring any positive results. Mathematics in the last century has experienced a brilliant growth in conjunction with the natural sciences. It should not be forgotten, however, that mathematics by its traditions and long history belongs to the liberal arts; it is certainly not to be regarded mainly as a technical tool of the sciences. There is an earnest desire among many humanistic students for information on the subject. They may not have the ability or time to partake in the regular, more technical courses. However, such a serious demand we shall have to satisfy even if it may mean a compromise in some of our traditional ways of instruction. This in the long run will undoubtedly be of as much benefit to mathematicians and the position of mathematics in our higher institutions of learning as to those for whom these concessions are made. And the instructors in mathematics can take pride if in this attempt some small bridge has been thrown over the chasm between the scientists and the humanists.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1943–44

Pi Mu Epsilon, University of Nebraska

This year six meetings were held, including business meetings, the initiation banquet, and the annual spring picnic. The following talks were given:

Invariants, by Professor M. G. Gaba

The isograph, by Mr. H. W. Linscheid

Gnomonic projections, by Professor O. C. Collins.

The annual Freshman and Sophomore competitions were held May 5, and the Freshman prize was won by Mr. Fay Fuchser. At the annual banquet held on February 29, eleven men were initiated. Because of wartime conditions, it was decided to invite members of the Army Specialized Training Program stationed at this University, who showed exceptional ability in mathematics, to become members of *Pi Mu Epsilon*. At the spring picnic on May 12, the pledge was administered to ten civilian and fifteen army students.

The officers elected for next year are: Director, Charles Lantz; Vice-Director, Margaret Stewart; Secretary, Mary Lou Weaver; Vice-Secretary, Noboru Tosaya; Treasurer, Mary Ann Mattoon; Vice-Treasurer, Anton Kashas. The faculty adviser is Professor F. S. Harper.

Kappa Mu Epsilon, Upsala College

The New Jersey Alpha chapter of *Kappa Mu Epsilon* concluded a very successful year with its fifth annual banquet, on June 2, at which two new members were initiated: Miss Audrey Richter and Mrs. Mary Louise McKim. At the banquet an interesting talk was given entitled

Cryptography, by Dr. D. R. Davis of Montclair State Teachers College.

Other topics and speakers for the year were:

Boolean algebra, by Elizabeth Ebel

The group concept, by Joseph Prieto

Duplication of a cube, squaring of the circle, and trisection of an angle, by Zelda Meisel

Practical applications of mathematics to engineering problems, by Dr. Fithian of Newark College of Engineering

Finite differences, by Betty Rudebock

Vector analysis, by Dorothy Shaw.

Officers for the coming year are to be: President Thales, Betty Rudebock; Vice-President Apollonius, Joseph Prieto; Secretary Abel, Elizabeth Ebel; Treasurer Fibonacci, Mrs. Mary Louise McKim; Historian Gauss, Marjorie Wolfe; Faculty Sponsor and Corresponding Secretary Descartes, Dr. M. A. Nordgaard.

Mathematics-Physics Club, College of Saint Teresa

The club numbered forty-five members who devoted the bi-monthly meetings to the study of aeronautics, which included the identification of planes according to the Wetfurn system, aerodynamics, and the engine; as well as the arithmetic, algebra, geometry, and trigonometry involved. A guest speaker discussed the bomber of the future. Slides and films illustrated certain topics. A Valentine party and a mathematical picnic conducted as a Naval Air Base comprised the social meetings of the year. Officers were: President, Dorothy Hilty; Vice-President, Josephine Schmelzle; Secretary, Mary Theo Maze; Treasurer, Dolores Raleigh; Faculty Advisors, Sister Thomas à Kempis, Sister Roswitha, Sister Leontius.

Mathematics Club, Illinois Institute of Technology

During the winter semester, the Mathematics Club met to hear papers read by various faculty members and students, as follows:

Elasticity, by Wally Ito

Transcendental numbers, by Dr. Herbert Busemann

Projective geometry, by Harley Flanders

Number theory, by Allen Devinatz

Binary system, by Felix Rosenthal

Brachistochrone problem, by Samuel Karlin

Laplace transformation, by Dr. W. B. Caton

Series summation, by Harley Flanders

Perfect numbers, by Samuel Karlin

Transformations of lineal elements, by Dr. John DeCicco

During the spring semester, bimonthly meetings were held, at which the following papers were presented:

Euler's summation formula and Stirling's formula for $n!$, by Samuel Karlin

Fourier's series, by Sidney Baker

Buffon's needle problem, by Gerald Rosenberg

Ordered systems—Lattices, by Dr. L. R. Wilcox

Continued fractions, by Alan Grant

Probability—Tchebycheff's problem, by Harley Flanders

Evaluation of $\int_0^\infty e^{-t^2}dt$ *and related forms*, by Berny Miretzky

Pascal's theorem and areas in plane geometry, by Don Slager

Interest was encouraged on the part of members in a broad range of topics, particularly in subjects not on the regular curriculum.

Officers for the winter semester were: President, Samuel Karlin; Secretary, Harley Flanders; Publicity, Morris Shapiro; Faculty Sponsor, Dr. H. Busemann. Officers for the spring semester were: President, Sidney Baker; Secretary, Harley Flanders; Faculty Sponsor, Dr. John DeCicco.

Kappa Mu Epsilon, Albion College

The 1943–44 program of the Michigan Alpha Chapter of *Kappa Mu Epsilon* included one meeting each month from October to May, and opened with a talk entitled

The history and purpose of Kappa Mu Epsilon, by K. R. Ferguson.

This was followed by biographical sketches by the following initiates: George Peacock, R. D. Barrow, Augustus DeMorgan, F. Aburano, George Boole Mari-Anne Gordon, James Sylvester, Beryl Voelker.

The November meeting was preceded by a roll call, calling for a *Recent mathematical article and comments*. Two talks were presented:

Special devices for integrating unusual forms, by J. R. Barcroft

Mathematics in war, by W. F. Voglesong.

Topics presented at the two joint meetings with the *Physics Club* in December and January were

Early world mathematics, by F. Aburano
Vectors applied to physics, by E. J. Dinger
Asymptotes oblique to the axes, by Betty Hossfeld
Some unusual applications to physics, by W. F. Voglesong.

Following the initiation at the February meeting, each of the new members gave a brief talk, and then all joined in a social hour with refreshments. Both the March and April meetings opened with a roll call. A feature of the April meeting was that seniors had complete charge of the program. Topics presented in March and April were

March roll call: *Continental mathematicians*
Homothetic properties and their application, by R. D. Barrow
Mathematical fallacies, by Virginia Tripp
April roll call: *An unusual application of mathematics*
Mathematics applied to physics, by E. J. Dinger and K. R. Ferguson
Mathematics applied to chemistry, by Betty Hossfeld and W. F. Voglesong.

The annual program was concluded with a joint picnic of the Mathematics, Physics, and Chemistry Clubs on May 8. Officers for the year 1943–44 were: President, K. R. Ferguson; Vice-President, J. R. Barcroft; Secretary-Treasurer, Betty Hossfeld; Member of Program Committee, Virginia Tripp.

Kappa Mu Epsilon, Louisiana State University

A total of fifty-eight members have been initiated during the year, at initiations which were held each quarter because of present conditions. Papers presented during 1943–44 were as follows:

Games of chance, by Dr. K. L. Nielsen
Base e, by Dr. Frank A. Rickey
The accident of ten fingers, by Dr. C. J. Thorne
The theorem of Desargues, by Miss Gloria McCarthy
Mathematics in nature, by Mr. Milton Peacock
Some notes on relativity, by Dr. Houston T. Karnes.

Each year the chapter gives two awards. An award of fifteen dollars, to the graduating senior who has proved superior according to the rules set up in the local chapter by-laws, was granted this year to Mr. R. J. Hoskins. A second award, conferred upon the freshman who ranks highest in a special *K.M.E.* examination, was granted to Mr. R. B. Stobaugh. The local chapter has obtained a conspicuous place in the Mathematics Library for the purpose of setting up a reading shelf of mathematical books of interest to students at large. Some twenty volumes have been placed on this shelf. The chapter also bought a one hundred dollar war bond in the Fourth Bond Drive.

Officers for 1944–45 are: President Gauss, Milton Peacock; Vice-President Poincaré, Mike Fossier; Secretary Fermat, Gene Ventre; Treasurer Galois, Yvonne Broussard; Historian Cajori, Albert Saliba; Faculty Sponsor, Marelena White; Secretary Descartes, Dr. H. T. Karnes.

Pi Mu Epsilon, University of Kentucky

Due to wartime conditions, and the presence of an Army Specialized Training unit at the University of Kentucky this year, *Pi Mu Epsilon* held only one meeting each quarter. The programs follow:

Problems in mathematical definitions, by Dr. H. H. Downing

The need for a mathematics for the social and biological sciences, by Professor Edward Newbury

Problems in inversion, by the following mathematics students: Mary Ann Mache, Virginia Baskett, Virginia Mitchell

Highlights from the history of Pi Mu Epsilon at University of Kentucky, presented by Dean P. P. Boyd, after the April initiation and dinner.

A picnic meeting was held at Needmore Farm in May. During the year, *Pi Mu Epsilon* voted to spend one hundred dollars for the purchase of books for the mathematics library. Officers for 1944–45 are: Director, Dr. Sallie Pence; Vice-Director, Mary Ann Mache; Secretary-Treasurer, Virginia Baskett; Librarian, Dr. L. W. Cohen.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

THE METHOD OF UNDETERMINED COEFFICIENTS

TOMLINSON FORT, Lehigh University

The method of undetermined coefficients for determining a "Particular Integral" in the theory of the linear differential equation with constant coefficients is explained in many text books, usually without proof of the rules that are announced. Proofs have been given by A. B. Coble,* also by Reddick and Miller† and possibly others. However, the following seems simpler. The use of Taylor's formula is rigorous and compact and a treatment of special cases is not necessary. The author has not seen a previous discussion for the difference equation.

1. The differential equation. Let us be given the differential equation with constant coefficients

$$F(D)y = (D^n + C_1D^{n-1} + \cdots + C_n)y = ke^{ax}x^m, \tag{1}$$

* A. B. Coble, Concerning a method for finding a particular integral, this MONTHLY, vol. 26, 1919, p. 12.

† Reddick and Miller, Advanced Mathematics for Engineers, p. 50.

where m is zero or a positive integer and a is any number real or complex. First establish the formula

$$D^j(e^{ax}\mu(x)) = e^{ax}(a + D)^j\mu(x). \tag{2}$$

This is readily done by mathematical induction.

From (2) it follows that

$$F(D)e^{ax}\mu(x) = e^{ax}F(a + D)\mu(x). \tag{3}$$

Rewrite (3) by the use of Taylor's formula utilizing well known properties of D

$$F(D)e^{ax}\mu(x) = e^{ax}\left(F(a) + F'(a)D + \frac{F''(a)}{2!}D^2 + \cdots + \frac{F^{(n)}(a)}{n!}D^n\right)\mu(x). \tag{4}$$

Now let us suppose that a is an r-fold root of the auxiliary equation

$$F(\alpha) = 0. \tag{5}$$

It is permitted that r be zero, that is that a be not a root. Assume a particular solution of the type

$$Y = (A_1x^{m+r} + A_2x^{m+r-1} + \cdots + A_{m+1}x^r)e^{ax}.$$

Substitute this in (1) and use (4) noting that $F(a) = F'(a) = \cdots = F^{(r-1)}(a) = 0$ since a is an r-fold root. We have

$$\begin{aligned} e^{ax}\Bigg[& x^mA_1\frac{(m + r)\cdots(m + 1)}{r!}F^{(r)}(a) \\ & + x^{m-1}\left(A_1B_{11} + A_2\frac{(m + r - 1)\cdots m}{r!}F^{(r)}(a)\right) \\ & + x^{m-2}\left(A_1B_{21} + A_2B_{22} + A_3\frac{(m + r - 2)\cdots(m - 1)}{r!}F^{(r)}(a)\right) \\ & + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & + x^0\left(A_1B_{m1} + A_2B_{m2} + \cdots + A_{m+1}\frac{r!}{r!}F^r(a)\right)\Bigg] = ke^{ax}x^m. \end{aligned}$$

Here the B's depend upon terms arising from operating with $D^{r+1}, \cdots, D^n$. Equate coefficients. Inasmuch as $F^{(r)}(a) \neq 0$ as a is only an r-fold root of (5), and inasmuch as $(m+r-j)\cdots(m+1-j) \neq 0$ since $r \geqq 0$ and $j \leqq m$ we can determine successively $A_1, A_2, \cdots, A_{m+1}$. The coefficients thus determined make Y a solution.

In case the right hand member is a sum of functions of the type $ke^{ax}x^m$ treat each such function separately. The work, of course, can be arranged to suit the convenience of the computer.

Suppose the right hand member contains a term of the form $ke^{ax}x^m \cos \lambda x$. Replace it by the sum of two terms of the respective types $k\frac{1}{2}x^me^{ax}(\cos \lambda x + i \sin \lambda x)$

and $k\frac{1}{2}x^m e^{\alpha x}(\cos \lambda x - i \sin \lambda x)$ or in more compact notation $k\frac{1}{2}x^m e^{\alpha x}$ and $k\frac{1}{2}x^m e^{\bar{\alpha} x}$. Denote the resulting particular solutions by Y_1 and $\overline{Y}_1$. If the coefficients in (5) are real, then if α is an r-fold root so is $\bar{\alpha}$. If k is also real the A's and the corresponding $\overline{A}$'s are conjugates. Consequently Y_1 and $\overline{Y}_1$ are conjugate and $Y = Y_1 + \overline{Y}_1$ is real. It is not, however, necessary to the scheme that coefficients in (5) be real. If the coefficients in (5) are real and if k and a are real one can work wholly in the domain of reals if he so desires. Assume the trial form for Y in the final known form:

$$Y = (A_1 x^{m+r} + A_2 x^{m+r-1} + \cdots + A_{m+1} x^r) e^{\alpha x} \cos \lambda x + (B_1 x^{m+r} + B_2 x^{m+r-1} + \cdots + B_{m+1} x^r) e^{\alpha x} \sin \lambda x.$$

We know that such a particular solution exists and consequently that the A's and B's are determinable. If the right hand member contains a term of the form $ke^{\alpha x} x^m \sin \lambda x$, modifications of our discussion are but slight and are left to the reader.

2. The difference equation. The work here is so similar to the above that it is only sketched. Suppose that we are given

$$F(E)y = K\beta^x x^{(m)},$$

where

$$E\mu(x) \equiv \mu(x+1) \equiv (1+\Delta)\mu(x).$$

The auxiliary equation is

$$F(\alpha) = 0.$$

Suppose that β is an r-fold root of this.

Assume a particular solution

$$Y = (A_1 x^{(m+r)} + A_2 x^{(m+r-1)} + \cdots + A_{m+1} x^{(r)})\beta^x.$$

We proceed to determine the A's as previously. Proof that this is always possible depends upon the formula

$$E^j(\beta^x \mu(x)) = \beta^x(\beta + \beta\Delta)^j \mu(x),$$

and consequently

$$F(E)\beta^x \mu(x) = \beta^x \left[F(\beta) + F'(\beta)\beta\Delta + \cdots + \frac{F^n(\beta)}{n!} \beta^n \Delta^n \right] \mu(x),$$

coupled with the fact that

$$\Delta x^{(j)} = j x^{(j-1)},$$

where

$$x^{(j)} = x(x-1)\cdots(x-(j-1)).$$

ON MY PROOF OF THE BUDAN-FOURIER THEOREM

M. F. SMILEY, Postgraduate School, U. S. Naval Academy

The author has given a proof of the Budan-Fourier theorem* which assumes a knowledge of the Fundamental Theorem of Algebra but is otherwise strictly elementary. The following theorem was assumed in our proof.

THEOREM. *If $f(x)$ is a polynomial with real coefficients and if α is a real number for which $f(\alpha)=0$, then there is a positive number ϵ such that*

$$f(x)f'(x) > 0, \qquad (\alpha < x \leqq \alpha+\epsilon),$$
$$f(x)f'(x) < 0, \qquad (\alpha-\epsilon \leqq x < \alpha).$$

Our proof of this theorem will rest on the even more fundamental

LEMMA. *If $f(x)$ is a polynomial with real coefficients and α is a real number for which $f(\alpha)>0$, then there is an interval $\alpha-\epsilon \leqq x \leqq \alpha+\epsilon$ on which $f(x)>0$.*

Remark. We might base a proof of our lemma on the fact that a polynomial is a continuous function. Such considerations are, however, not essential.

Proof of the lemma. Since $f(x)$ has real coefficients, its complex roots occur in conjugate imaginary pairs, and we have

$$f(x) = a_0(x-r_1)(x-r_2)\cdots(x-r_m)(x^2+2a_1x+b_1)\cdots(x^2+2a_px+b_p)$$

in which each r_j $(j=1, \cdots, m)$ is real, $r_1 \leqq r_2 \leqq \cdots \leqq r_m$, and each factor $x^2+2a_ix-b_i$ $(i=1, \cdots, p)$ arises from a pair of conjugate complex roots. Each of the factors $x^2+2a_ix+b_i$ is *positive* for all real x since $a_i^2-b_i<0$ and $x^2+2a_ix+b_i=(x+a_i)^2+b_i-a_i^2>0$. For convenience, we set $r_0=-\infty$, $r_{m+1}=+\infty$. Since $f(\alpha)>0$, we have $r_j<\alpha<r_{j+1}$ for some $j=0, \cdots, m$. Define ϵ to be the smaller of $|r_{j+1}-\alpha|/2$ and $|r_j-\alpha|/2$. We see that no factor of $f(x)$ can change sign on the interval $\alpha-\epsilon \leqq x \leqq \alpha+\epsilon$. Since $f(\alpha)>0$, the proof is complete.

Proof of the theorem. Using the Fundamental Theorem of Algebra, we may write $f(x)=(x-\alpha)^\mu\phi(x)$, in which μ is a positive integer and $\phi(\alpha)\neq 0$. We calculate† $f'(x)=\mu(x-\alpha)^{\mu-1}\phi(x)+(x-\alpha)^\mu\phi'(x)$. We then have

$$f(x)f'(x) = (x-\alpha)^{2\mu-1}\{\mu[\phi(x)]^2+(x-\alpha)\phi(x)\phi'(x)\}.$$

Setting $g(x)$ equal to the polynomial inside the parentheses, we note that $g(\alpha)=\mu[\phi(\alpha)]^2>0$. Our theorem now follows immediately from our lemma.

We take this opportunity to correct two typographical errors in our original proof. The equations $g'(\alpha)=g''(\alpha)=\cdots=g^{(\nu+1)}(\alpha)=0$ should be replaced by $g'(\alpha)=g''(\alpha)=\cdots=g^{(\nu)}(\alpha)=0$. In the sentence beginning "If ν is even $\cdots$" replace $g^{(\nu)}(x)$ by $g^{(\nu+1)}(x)$.

* An inductive proof of Budan's theorem, this MONTHLY, vol. 49, 1942, pp. 112–113. *Cf.* N. B. Conkwright, An elementary proof of the Budan-Fourier theorem, this MONTHLY, vol. 50, 1943, pp. 603–605.

† Here we use the formula $(fg)'=f'g+fg'$ for *polynomials*. This may be proved *algebraically* as well as by the standard methods of the calculus.

A REPRESENTATION OF INTEGERS

MICHAEL WILENSKY, Cincinnati, Ohio

The solution for q of the two equations

$$q^2 - r = km, \tag{1}$$

$$(m - q)^2 - r = q'^2 - r = k'm, \tag{2}$$

where $(q, m) = 1$; $q < m/2$, $m > q' > m/2$, and therefore $k < m/4$, $k' > m/4$; $|r| < m$ and not a square, gives us $2q = m - (k' - k)$. By substituting this value of q in (1), we obtain the relation

$$m^2 - 2m(k' - k) + (k' - k)^2 - 4r - 4km = 0.$$

The solution of this quadratic equation for m gives us the representation

$$m = k + k' + 2\sqrt{kk' + r}. \tag{3}$$

(The minus sign before the radical should be omitted, for $q \geqq k$, $q' \geqq k'$ and $m = q + q' \geqq k + k'$.) The integer m is thus represented as a function of k, k', and r. Since every pair of integers which satisfies the congruence $x^2 \equiv r \pmod{m}$ leads to a representation (3), there are $2^{\lambda - 1}$ representations of an integer m for a given r, if we denote by 2^λ the number of the distinct roots of the congruence. Particularly in case m is an odd prime or double an odd prime, when the congruence has two distinct roots, the representation of m for a given r is *unique*.

It must, however, be borne in mind that two congruent residues modulo m lead to *different* representations of this number, and that *every* r has a congruent quadratic residue within the interval we set for it: $|r| < m$, *viz.*, $r > 0$ and $-(m - r)$. If the representation which corresponds to r has the form of $m = k + k' + 2\sqrt{kk' + r}$, the representation by the other residue will be

$$m = (k + 1) + (k' + 1) + 2\sqrt{(k + 1)(k' + 1) - (m - r)}.$$

Representation (3) characterizes the integer m, for conversely, if an integer m has a representation (3), it has r as a quadratic residue. In fact, multiplying both sides of (3) by k, we obtain

$$(k + \sqrt{kk' + r})^2 - r = km.$$

Comparing this expression with (1) we have

$$q = k + \sqrt{kk' + r}. \tag{4}$$

Similarly multiplication of (3) by k' and comparison with (2) gives us

$$q' = k' + \sqrt{kk' + r}, \tag{4a}$$

so that (3) is the sum of (4) and (4a).

From (3) it is also evident that if m is odd, k and k' are of different parity, and if it is even, both are of the same parity.

The following table illustrates (3) for $r = -1$.

m	q	k	k'	$2\sqrt{kk'-1}$
13	5	2	5	6
$65=5.13$	8	1	50	14
"	18	5	34	26
$1105=5.13.17$	47	2	1013	90
"	242	53	674	378
"	268	65	634	406
"	463	194	373	538

We shall now consider only the special case, $r=-1$. As it is well known, only a number the prime divisors of which are congruent to 1 modulo 4, or double such a number, has minus 1 as a quadratic residue and that such a number can be represented as a sum of two squares $m=a^2+b^2$, where we may suppose $(a, b)=1$. It is also known that one can always find two relatively prime numbers $x \leqq a$ and $y \leqq b$ which satisfy the equation $ay-bx=\pm 1$. Putting $a=x_1+x_2$, $b=y_1+y_2$ and letting x and y assume the values x_2, y_2, respectively, we have $(x_1+x_2)y_2-(y_1+y_2)x_2=\pm 1$, or

$$x_1y_2 - y_1x_2 = \pm 1. \tag{5}$$

Since k and k' have, like m, minus 1 as a quadratic residue, each of them can also be represented as a sum of two squares. We assume $k=x_1^2+y_1^2$, $k'=x_2^2+y_2^2$ and write (3) accordingly

$$m = a^2 + b^2 = (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y^2) - 1}. \tag{6}$$

Now

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 + y_1y_2)^2 + (x_1y_2 - y_1x_2)^2.$$

The last expression, according to (5), equals 1. Hence (6) may be written $m=(x_1+x_2)^2+(y_1+y_2)^2$; our assumption is thus proved. Accordingly

$$q = k + \sqrt{kk' - 1} = x_1(x_1 + x_2) + y_1(y_1 + y_2) = ax_1 + by_1, \tag{7}$$

and

$$q' = x_2(x_1 + x_2) + y_2(y_1 + y_2) = ax_2 + by_2. \tag{7a}$$

If x_1 and y_1 are given fixed values α, γ, respectively, so that $k=x_1^2+y_1^2$ is a fixed number, and moreover, if β, δ are the smallest integers x_2, y_2, respectively, which satisfy (5) and (3) or (6) so that $k<m/4$, all integers m having the same k can be represented as a function of t:

$$m = f(t) = (\alpha + \beta + \alpha t)^2 + (\gamma + \delta + \gamma t)^2,$$

for $\alpha(\delta+\gamma t)-\gamma(\beta+\alpha t)=\alpha\delta-\beta\gamma=\pm 1$. Expanding, we have

$$(8)\qquad \begin{aligned} f(t) &= (\alpha^2+\gamma^2)t^2 + 2[(\alpha^2+\gamma^2)+\alpha\beta+\gamma\delta]t + (\alpha+\beta)^2+(\gamma+\delta)^2 \\ &= kt^2 + 2q(0)t + f(0), \end{aligned}$$

the determinant of which, $[q(0)]^2-kf(0)$, is indeed minus 1. We can thus represent the entire class of integers having the same k by the binary quadratic form

$$kt^2 + 2q(0)tu + f(0)u^2,$$

when α, β, γ, δ are given, in putting $u=1$ and giving t all positive values, including 0. To every positive value of t corresponds a different value of m.

From (8) we learn that

$$(9)\qquad f(t+1) = f(t) + k + 2[kt + q(0)].$$

Since $kf(t)-1=[kt+q(0)]^2$, we can write (9) in the form

$$(9a)\qquad f(t+1) = k + f(t) + 2\sqrt{kf(t)-1}.$$

Regarding k' as a function of t and comparing (9a) with (3), we have $k'(t+1)=f(t)$. Now the derivative

$$(10)\qquad f'(t) = 2[kt + q(0)] = 2\sqrt{kf(t)-1}.$$

Therefore $f(t+1)=k+f(t)+f'(t)$. Since $[q(t)]^2=kf(t)-1$, we have $q(t)=\frac{1}{2}f'(t)$, and

$$(11)\qquad q(t+1) - q(t) = \tfrac{1}{2}[f'(t+1) - f'(t)] = k.$$

The q of the integers m with the same k thus form an arithmetic progression with the difference k, whereas m, q', and k'—each form a sequence with the second difference $2k$.

This line of reasoning, that r is also a quadratic residue of k, is applicable to every r. An expression of the form $kt^2+2q(0)t+f(0)$ with the determinant r represents the class of integers having r as a quadratic residue and k as coefficient.

Let us give some examples for $r=-1$:

Example I. Let $k=1$, then $\alpha=0$, $\beta=1$, $\gamma=1$, $\delta=1+t$; $m=(t+2)^2+1=t^2+4t+5$; $q=\frac{1}{2}f'(t)=t+2$; $k'=(t+1)^2+1$.
Putting $x=t+2$, we have $m=x^2+1$, $x\geqq 2$.

Example II. Let $k=5$. There are two classes:

1. $\alpha=1$, $\beta=1+t$, $\gamma=2$, $\delta=3+2t$; $\alpha\delta-\beta\gamma=1$; $m=(t+2)^2+(2t+5)^2=x^2+(2x+1)^2$, $x\geqq 2$.

2. $\alpha=1$, $\beta=2+t$, $\gamma=2$, $\delta=3+2t$; $\alpha\delta-\beta\gamma=-1$; $m=(t+3)^2+(2t+5)^2=x^2+(2x-1)^2$, $x\geqq 3$.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University, and H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editors of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Basic Mathematics for War and Industry. By P. H. Daus, J. M. Gleason, and W. M. Whyburn. New York, The Macmillan Company, 1944. 11+277 pages. $2.00.

This text provides in a single volume topics of elementary mathematics which are usually found in separate text-books. The book begins with a short chapter, 28 pages, on arithmetic which includes common and decimal fractions, approximations and significant figures, rules for extracting square root, and mensuration. This is followed by a chapter on each of the following topics: algebra, plane geometry, plane trigonometry, solid geometry, and spherical trigonometry. The book includes a table of squares, cubes, square roots, and cube roots of the integers 1 to 100; a table of four place logarithms, a table of four place trigonometric functions, both natural and logarithmic; and answers to all but a very few of the over 500 exercises which are given in the 59 sets. There is an adequate index as well as a very detailed table of contents. A paper protractor is supplied with each book.

The exposition is surprisingly detailed when one considers how much material is covered in the book. The typography is clear and the numerous figures are excellent. As might be expected in a condensed text, many proofs are omitted and some propositions from geometry are called axioms. However, some of the brevity is attained by presenting clever short proofs and by avoiding the formal superposition and statement-reason types of proof. The illustrative problems and the exercises throughout the book draw upon all fields of applications with special emphasis on aerial and naval navigation, physics, and geometry. An exceptional feature is the use of spherical trigonometry and spherical geometry in the solution of shop problems.

The authors state in the preface, "The book should be a suitable text for courses offered under the Engineering, Science, and Management War Training Program; for the various armed service courses, both within army schools and in colleges; for vocational and other specialized school courses." Furthermore, the reviewer agrees with the authors that the book is "thoroughly practical" and that the arrangement and treatment of the topics would permit selection of the special parts that a course may require.

LOUISE A. WOLF

Intermediate Differential Equations. By E. D. Rainville. New York, John Wiley and Sons, Inc., 1943. 6+213 pages. $2.75.

This text serves a useful purpose in providing a second course in differential equations for students who have finished a first year elementary course. Possibly a more descriptive title would be *Linear Differential Equations of Second Order*, since the treatment, except for Chapter II on the Riccati equation and a brief excursion in Chapter X, is devoted almost exclusively to this type.

The purpose of the text as set forth in Chapter I is three-fold. It is to bring into play certain mathematical tools useful in the solution of linear ordinary differential equations; to train the student to grasp the aims, methods and accomplishments of the theory so that he may proceed to advanced work; and to present material of interest for its own sake. After two chapters in which the linear equation of second order is introduced and attacked in an elementary manner, including an approach through the Riccati equation, the third chapter treats briefly of selected complex variable theory suitable for making use of series solutions. Chapter IV then gives the technique of power series solutions with numerical examples. This leads naturally to a treatment of equations of the Fuchsian type, to the hypergeometric equation and to Kummer's twenty-four solutions.

Chapters VII and VIII develop the method of confluence of singularities with the Fuchsian equation as a point of departure. Series solutions are obtained for Bessel's equation and for Whittaker's confluent hypergeometric equation. Chapter IX treats of some classical equations, and the final chapter gives a few applications, such as: a fourth order equation used in the design of large pipes; a problem of Timoshenko in deformation of circular plates; temperature in an infinite solid cylinder, with a brief word on other boundary value problems.

One question that naturally comes to mind is whether there are other ways of orienting an intermediate course. For example, one could take typical cases of differential equations from the engineering and physical applications, and develop one or more theories around each type case. Such a method might appeal strongly to engineers and physicists. Also, such a scheme might have the added advantage of flexibility: *e.g.*, the instructor could choose to go into operational theory, say (including the more modern phases of it), as far as the capabilities of the students and the allotted time permitted.

The treatment throughout is lucid and well illustrated with problems. The student who has had a first course will find this an excellent preparation for the more weighty discussions of advanced theory. Those who need a better command of differential equations than they obtain in an elementary course, can profit by reading the text and working the problems.

T. M. Simpson
B. F. Dostal

Principles of Air Navigation. By B. A. Shields. New York and London, McGraw-Hill Book Company, Inc., 1943. 7+451 pages. $2.20.

This book is the third in a series on introductory aviation. It is written primarily to be readable by secondary school students. Yet, at the same time it gives a very interesting account of the intimate problems of pilots and navigators. The range of its content is perhaps best shown by listing the titles of the chapters. These are nine in number, as follows:

1. Maps and Charts.—2. Map Reading.—3. Cross-Country Navigation.—4. Dead-Reckoning Navigation.—5. Practical Navigation Problems.—6. Radius of Action.—7. Computers and Calculators.—8. Radio Navigation.—9. Over-Water Navigation.

The first few chapters deal with the various types of maps, special discussion being given to aviation charts and special aeronautical charts. The interpretation of charts and maps is very carefully considered.

The next few chapters deal with the use of the compass and other navigation instruments in laying out the course, calculating the drift, solving problems of windage, and determining air speed. There then follows a very fine discussion on the factors which limit the radius of action in combat.

Chapter 7 treats primarily of the various types of calculators for solving graphically the usual problems of air navigation.

Chapter 8 is devoted to radio communications, signal systems, range finding, and radio beam navigation.

Chapter 9, which is entitled "Over-Water Navigation," considers only the fundamental aspects of this type of navigation. Many of the problems which have been discussed earlier in the text are here solved on a Mercator plotting sheet. The real problem of celestial navigation is discussed on pages 409–446. This is mainly an explanatory discussion, not complicated by the theory necessary for understanding fully such problems. This latter comment does not infer a lack of thoroughness on the part of the author, but is intended to show that he thoroughly understood the background of his probable readers and did not intend that this book should constitute a text on celestial navigation.

Throughout the book there are numerous reproductions of maps, drawings and photographs of instruments, charts showing the graphical solution of problems, and all of the work of illustrations is remarkably well done. The entire book is remarkably well written, and the reviewer recommends it highly to those who would approach the topic of aviation from a practical and elementary point of view.

G. E. MOORE

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 24 Strathearn Boulevard, Toronto 10, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 636. *Proposed by P. R. Halmos, Syracuse University*

What is the least number of links that have to be cut, in a chain of n links, so that every integer between 1 and n can be represented as a sum of numbers of links in the disconnected chains obtained? (Cutting one link which is not at either end of a connected chain produces three pieces: the severed link and the two ends which it held together.)

E 637. *Proposed by V. Thébault, San Sebastián, Spain*

Locate a plane which touches four equal spheres inscribed in the trihedra at the respective vertices of a given tetrahedron.

E 638. *Proposed by C. H. Wolfe, Lakeside High School, Ohio*

Without the use of tables, find the smallest integer whose cube terminates in seven sevens.

E 639. *Proposed by Howard Eves, Syracuse University*

A *clothoid* (or *transition spiral*, used in highway engineering) is defined as a curve whose curvature varies directly with the arc length. Locate the geometrical pole of this spiral.

E 640. *Proposed by E. D. Schell, Arlington, Virginia*

Solve in integers the equation

$$x^y = y^x \qquad (x > y).$$

SOLUTIONS

Arrangements in a Circle

E 599 [1943, 634]. *Proposed by U. P. Davis, University of Florida*

In how many different ways can $3n$ objects be arranged in a circle, if there are n different kinds, 3 of each kind?

I. *Solution by J. B. Kelly, Princeton University.* Let any object of the kth kind be denoted by a_k; $k=1, \cdots, n$. We first determine the number of ways of arranging the $3n$ objects in a row with a particular object a_1 in the initial position; this is

$$\frac{(3n-1)!}{2!(3!)^{n-1}}.$$

Now separate out the $(n-1)!$ permutations of the form

$$a_1a_{i_2}a_{i_3}\cdots a_{i_n}a_1a_{i_2}a_{i_3}\cdots a_{i_n}\cdots a_1a_{i_2}a_{i_3}\cdots a_{i_n}.$$

For every permutation P not of this form there are two others, distinct from P, having a_1 in the initial position, and obtainable from P by cyclic permutation. Hence the required number of permutations is

$$\frac{1}{3}\left(\frac{(3n-1)!}{2\cdot 6^{n-1}}-(n-1)!\right)+(n-1)!=\frac{(3n-1)!}{6^n}+\frac{2}{3}(n-1)!.$$

The corresponding problem with 3 replaced by m is quite difficult, and seems to involve the decomposition of m into factors.

II. *Note by E. P. Starke, Rutgers University.* The required number of arrangements is

$$2\{(n-1)!\}/3+(3n-1)!/6^n.$$

This is a special case of formulas developed by H. S. Grant in connection with his solution of problem no. 519 in the *National Mathematics Magazine*, April, 1944. He solves there the more general problem of finding the number of cyclic permutations of $x_1+\cdots+x_n$ letters of which x_1 are alike, x_2 others are alike, and so on. The present problem merely requires that we put $x_1=\cdots=x_n=3$.

Also solved by R. G. Blake, D. H. Browne, E. D. Schell, and the proposer. Blake and Browne give the expression

$$(p-1)\{(n-1)!\}/p+(pn-1)!/(p!)^n$$

for the case when $x_1=\cdots=x_n=p$, a prime. Schell refers to R. E. Allardice, *Proceedings of the Edinburgh Mathematical Society*, vol. 8, 1890.

A Square and a Cube with Ten Different Digits

E 602 [1944, 46]. *Proposed by V. Thébault, San Sebastián, Spain*

With the ten different digits, taken once each, form two numbers which are respectively the square and the cube of two consecutive multiples of 3.

Solution by Monte Dernham, San Francisco. In order that exactly ten digits be utilized, the integer to be cubed cannot be less than 48 nor greater than 99. Of the 18 multiples of 3 within this range, the cubes of only five, *viz.*, 66, 69, 75, 84 and 93, consist wholly of unrepeated digits. The first four of these must be discarded because, in each instance, both the square of the preceding and of the succeeding multiple of 3 would, if employed, involve a repetition of digits.

The square of 90 is similarly inadmissible. Hence the only solution is

$$9216 = 96^2, \qquad 804537 = 93^3.$$

Also solved by Colin Blyth, D. H. Browne, W. E. Buker, Walter Penney, E. D. Schell, E. P. Starke, Alan Wayne, and the proposer.

A Symmetrical Set of Five Points

E 605 [1944, 47]. *Proposed by L. M. Kelly, U. S. Coast Guard Academy*

If a set of four coplanar points has the property that the circumcircles of all subsets of three are equal (but not coincident), then the set is orthocentric. (See R. A. Johnson, *Modern Geometry*, p. 75.) Establish the existence of an analogous set of five points in space, *i.e.*, such that the circumspheres of all subsets of four are equal (but not coincident).

Solution by the Proposer. In his solution to E 540 [1943, 389], Eves established the existence of a tetrahedron whose sixteen-point sphere has a radius equal to one-half the circumradius. The vertices of such a tetrahedron, together with the isogonal conjugate of its circumcenter, form a set with the prescribed properties. For, the reflected images of this isogonal conjugate in the four faces can be shown to lie on the circumsphere. This follows from the fact that the images all lie on a sphere whose center is the circumcenter and whose radius is twice that of the sixteen-point sphere. (See N. A. Court, *Modern Pure Solid Geometry*, p. 244.) Thus the four spheres through the isogonal conjugate and the triads of vertices are all equal to the circumsphere.

Homothetic Orthocentric Tetrahedra

E 607 [1944, 88]. *Proposed by V. Thébault, San Sebastián, Spain*

Consider an orthocentric tetrahedron $ABCD$, of orthocenter H. Let O, A', B', C', D' be the circumcenters of the tetrahedra $ABCD$, $BCDH$, $CDAH$, $DABH$, $ABCH$. Prove that the tetrahedra $ABCD$ and $A'B'C'D'$ are homothetic from a center which divides OH in the ratio 3:2. Show also that the lines AA', BB', CC', DD' pass through the centers of gravity of the respective tetrahedra $BCDH$, $CDAH$, $DABH$, $ABCH$.

Solution by the Proposer. Let λ denote the power of H with respect to the circumsphere O. Then, if A'', B'', C'', D'' are the feet of the altitudes, we have

$$HA \cdot HA'' = HB \cdot HB'' = HC \cdot HC'' = HD \cdot HD'' = \lambda/3,$$

whence $OA' = \lambda/2HA'' = \frac{3}{2}HA, \cdots$ (See Thébault, *Annales de la Société Scientifique de Bruxelles*, 1924, p. 176). The tetrahedra $A'B'C'D'$ and $ABCD$ are thus inversely homothetic, in the ratio $-3/2$, and the homothetic center divides OH in the ratio 3/2.

The last part of the problem can be generalized as follows. For any point P, the lines joining the vertices A, B, C, D to the centroids of the respective tetrahedra $BCDP$, $CDAP$, $DABP$, $ABCP$ are concurrent.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4132. *Proposed by T. H. Matthews, McGill University*

If an aircraft travels at a constant airspeed, and traverses (with respect to the ground) a closed curve in a horizontal plane, the time taken is always less when there is no wind, than when there is any constant wind.

4133. *Proposed by A. L. Putnam, Yale University*

Let a, b, and c be integers with $b \neq 0$, and let d and f be the respective greatest common divisors of a and b and c and b. Then if

$$a \not\equiv \pm d \pmod{b} \quad \text{and} \quad d \not\equiv \pm f \pmod{b},$$

there is an infinite number of integers k for which the equation

$$ax + bxy + cy = k$$

has no solutions in integers.

4134. *Proposed by Hüseyin Demir, Columbia University*

Let $C_1^1C_2^1C_3^1$ be the inscribed triangle of a reference triangle $A_1A_2A_3$, and $C_1^2C_2^2C_3^2$ be that of $C_1^1C_2^1C_3^1$, and so on, obtaining a triangle $C_1^nC_2^nC_3^n$ after n steps. Denoting the angles of the nth triangle by C_i^n, prove that

1. $(C_i^n - \pi/3)/(A_i - \pi/3) = (-1)^n 2^{-n}$.

2. The limit of the direction of $C_2^nC_3^n$ as $n \to \infty$, is the direction of one of the trisectrices of the angle $(A_2A_3, C_2^1C_3^1)$, and from that observe a method of trisecting an angle by ruler and compass in infinitely many steps.

4135. *Proposed by V. Thébault, San Sebastián, Spain*

With the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken one at a time, form two numbers of five digits each such that one is a multiple of the other.

SOLUTIONS

A Definite Gamma Integral

4066 [1943, 65]. *Revised. Proposed by Richard Bellman, Brooklyn College*

Prove that

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \int_0^1 \left[1 + \frac{e}{x} - \frac{e}{1!(x+1)} + \frac{e}{2!(x+2)} - \cdots\right] \frac{dx}{\Gamma(x)}.$$

Solution by the Proposer. We have

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \sum_{n=1}^{\infty} \int_{n-1}^{n} \frac{dx}{\Gamma(x)}. \tag{1}$$

In the nth integral, $n \geqq 2$, set $x' = x-(n-1)$, and using the relation $\Gamma(x+1) = x\Gamma(x)$, this integral becomes

$$\int_0^1 \frac{dx'}{x'(x'+1)\cdots(x'+n-2)\Gamma(x')}$$

and we have, after replacing x' by x

$$\int_0^\infty \frac{dx}{\Gamma(x)} = \int_0^1 \left[1 + \frac{1}{x} + \frac{1}{x(x+1)} + \frac{1}{x(x+1)(x+2)} + \cdots\right] \frac{dx}{\Gamma(x)}.$$

Using Prym's identity for the infinite series in the brackets, we obtain the desired result.

Editorial Note. The Prym's identity may be established by the method of finite differences, or by the resolution of the fractions $1/x(x+1)\cdots(x+n)$ into partial fractions. The summation is then performed in a different order, and this change may be justified.

The Euler Function

4068 [1943, 65]. *Proposed by James Singer, Brooklyn College*

If a and n are integers, each greater than unity, and r is any positive integer, and if $\phi(x)$ is the Euler function, show that $\phi(1+a^r+a^{2r}+\cdots+a^{nr})$ is a multiple of $2r(n+1)$.

Solution by N. G. Gunderson, Cornell University. Let $1+a^r+a^{2r}+\cdots+a^{nr} = (a^{r(n+1)}-1)/(a^r-1) = A$. Then by Theorems I and V of the paper, *On the Integral Divisors of* a^n-b^n by G. D. Birkhoff and H. S. Vandiver on page 173 of the Annals of Math., series II, vol. 5, 1903, there exists a prime divisor p of A such that $p \equiv 1 \bmod r(n+1)$, except when $r=n=a=2$. But $2\cdot 2\cdot 3 \mid \phi(1+2^2+2^4)$.

If both r and $n+1$ are odd, $p \equiv 1 \bmod 2r(n+1)$, and so, since $(p-1) \mid \phi(A)$, we have that $2r(n+1) \mid \phi(A)$. If $n+1$ is odd and r is even, then $A = [(a^{r(n+1)/2}-1)/(a^{r/2}-1)]\,[(a^{r(n+1)/2}+1)/(a^{r/2}+1)]$. If $n+1$ is even, then $A = [(a^{r(n+1/2)}+1)]\,[(a^{r(n+1/2)}-1)/(a^r-1)]$. In each case each of these factors of A is greater that 2, and their greatest possible common divisor is 2. Hence either there exists an odd prime divisor q of A different from p, in which case $r(n+1)(q-1) \mid \phi(A)$, so that $2r(n+1) \mid \phi(A)$, or else $2^\alpha \cdot p \mid A$ where $\alpha > 1$, and again $2r(n+1) \mid \phi(A)$.

Note. A paper by Gunderson entitled *Some Theorems on the Euler ϕ-Function* in the Bulletin of the American Mathematical Society, vol. 49, 1943, pp. 278–280 contains the proof of several facts closely related to this problem.

Tetrahedron of Excenters

4069 [1943, 65]. *Proposed by V. Thébault, San Sebastián, Spain*

Let I_a, I_b, I_c, I_d denote the centers of the spheres escribed in the truncated trihedral angles for the corresponding vertices of the tetrahedron $ABCD$, and A_1, B_1, C_1, D_1 the points where the straight lines AI_a, *etc.* meet the faces BCD, *etc.* Show that

(1) $V'/V = -ABCD/(S-A)(S-B)(S-C)(S-D)$,

(2) $V_1/3V = -ABCD/(B+C+D)(C+D+A)(D+A+B)(A+B+C)$,

where V, V', V_1 denote the volumes of the tetrahedrons $ABCD$, $I_aI_bI_cI_d$, $A_1B_1C_1D_1$; A, *etc.* denote the areas of the faces BCD, *etc.* and $2S=A+B+C+D$.

Note. The formula (2) is due to Genty, *Nouvelles Annales de Mathématiques*, 1880, p. 528 and 1881, p. 341.

Editorial Note. The proposer gave the following information: The straight lines AI_a, BI_b, CI_c, DI_d are concurrent in I, the incenter of the tetrahedron $ABCD$; and, since the vertices of the latter are external centers of similitude of the pairs of the inscribed sphere (I) with the escribed spheres (I_a), (I_b), $)I_c)$, (I_d), we have four equations such as

$$(1) \qquad II_a = -\frac{(r_a - r)}{r} IA,$$

where r, r_a, r_b, r_c, r_d are the respective inradius and four exradii. There are four equations such as

$$(2) \qquad \frac{\text{vol. } IBCD}{\text{vol. } II_bI_cI_d} = -\frac{IB \cdot IC \cdot ID}{II_b \cdot II_c \cdot II_d},$$

which lead to four corresponding equations

$$(3) \qquad V_a = \text{vol. } II_bI_cI_d = -8A \cdot B \cdot C \cdot Dr r_b r_c r_d/81V^3,$$

where $V' = V_a + V_b + V + V_d$. Then

$$(4) \qquad V' = -16A \cdot B \cdot C \cdot Dr_a r_b r_c r_d/81V^3,$$

from which the desired equation for V'/V results.

Since the lines AI, BI, CI, DI pass respectively through A_1, B_1, C_1, D_1, it suffices to proceed similarly to obtain

$$(5) \qquad V_1/3V = -A \cdot B \cdot C \cdot D/(2S-A)(2S-B)(2S-C)(2S-D).$$

which is the second desired result. The equality (5), as stated in the problem

note, is due to Genty; and it was deduced in a manner different from the above by Faure, *l. c.* 1881, p. 341.

Since the derivations are rather tedious if carried out in a random manner, we shall give some formulas which may be used to shorten the labor:

$$3V = 2rS = 2r_a(S - A) = 2r_b(S - B) = \cdots;$$

$$(6) \qquad \frac{r - r_a}{r} = -\frac{2}{3}\frac{Ar_a}{V}; \qquad \frac{II_a}{IA} = -\frac{2}{3}\frac{Ar_a}{V} = -\frac{A}{S - A};$$

$$\text{vol. } IBCD = rA/3.$$

The first line of equations give the known result

$$\frac{2}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} + \frac{1}{r_d}.$$

The equations (6) with (2) give

$$V_a = -\frac{A \cdot B \cdot C \cdot Dr(S - A)}{3(S - A)(S - B)(S - C)(S - D)},$$

and from this we find at once the desired result for V'/V.

We now consider V_1 and we have equations of the type

$$\frac{IA_1}{IA} = \frac{r - r_a}{r + r_a} = -\frac{A}{(2S - A)};$$

$$\frac{\text{vol. } IB_1C_1D_1}{\text{vol. } IBCD} = \frac{IB_1 \cdot IC_1 \cdot ID_1}{IB \cdot IC \cdot ID} = -\frac{B \cdot C \cdot D}{(2S - B)(2S - C)(2S - D)};$$

$$\text{vol. } IB_1C_1D_1 = -\frac{rA \cdot B \cdot C \cdot D(2S - A)}{3(2S - A)(2S - B)(2S - C)(2S - D)}.$$

This gives at once the desired expression for $V_1/3V$.

The triangle ABC may be considered in the same way. If S and S' denote the areas of the triangle ABC and of $I_aI_bI_c$, we have

$$\frac{S'}{S} = \frac{abc}{2(s - a)(s - b)(s - c)}, \qquad 2s = a + b + c,$$

where a, b, c are lengths of the sides. This gives

$$S' = 2Rs,$$

where R is the circumradius.

Orthological Tetrahedrons

4075 [1943, 203]. *Proposed by N. A. Court, University of Oklahoma*

If the radical center of four spheres coincides with the Monge point of the tetrahedron (T) determined by their centers, the tetrahedron (S) formed by the

four radical planes of the given spheres with their orthogonal sphere is orthological to the twin tetrahedron (T') of (T) (*i.e.*, the perpendiculars dropped from the vertices of (S) upon the corresponding faces of (T') are concurrent).

Solution by the Proposer. The radical plane α of the given sphere (A) and the orthogonal sphere (M) of the four given spheres (A), (B), (C), (D) is perpendicular to the line AM joining the centers A, M of (A), (M). Now if M is the Monge point of the tetrahedron $(T)=ABCD$, the line AM is parallel to the line OA' joining the circumcenter O of (T) to the vertex A' of the twin tetrahedron (T') which corresponds to the vertex A of (T).

Thus the perpendicular from A' upon α passes through (O), and it may be shown in a like manner that the same holds for the other vertices of (T'). Hence the perpendiculars from the vertices of the tetrahedron (S) upon the corresponding faces of (T') are also concurrent (see the proposer's *Modern Pure Solid Geometry*, p. 147, art. 460. Macmillan, 1935).

Editorial Note. Since T' is the twin tetrahedron for T, their common centroid G is their homothetic center with ratio $(1:-1)$. Let (M) be any chosen sphere with arbitrarily chosen center M and radius r; then the polar reciprocal of T with respect to (M) is a tetrahedron S, and T and S are orthological with M as the common point of concurrency of the two sets of perpendiculars. In the homothetic transformation $T\leftrightarrow T'$, $S\leftrightarrow S'$ with corresponding faces parallel, and $(M)\leftrightarrow(M')$. Hence the perpendiculars from the vertices of T' to corresponding faces of S are concurrent in M', whereas those from vertices of S to corresponding faces of T' are concurrent in M. For the special case where M is the Monge point of T, the point M' is the circumcenter of T.

Prime Integers

4083 [1943, 330]. *Proposed by P. Erdös, Princeton, N. J.*

Let $a_1<a_2<\cdots<a_x\leqq n$ be an arbitrary sequence of positive integers such that no a_i divides the product of the others, then $x\leqq\pi(n)$, where $\pi(n)$ denotes the number of primes not exceeding n.

Solution by Whitney Scobert, University of Oregon. Any number $\leqq n$ is a product of powers of one or more of the $\pi(n)$ primes which are $\leqq n$. In order that this number not divide the product of the others it is necessary that it have at least one prime factor raised to a higher power than the sum of the powers to which it is raised in all of the other numbers of the sequence. This implies that it must be to a higher power than it is in any other number of the sequence. Since any prime can have its greatest power in only one of the numbers this allows for at most $\pi(n)$ numbers in the sequence.

In this connection, there is a statement on page 4 of Hardy and Wright's *Theory of Numbers* which has puzzled me for several years. The book states that there are 50,847,478 primes less than one billion and that these are not individually known. I would like to have a reference to the method of obtaining

this figure. It is used on page 9 to check the asymptotic prime number theorem so that evidently it was not found by the theorem.

Solved also by S. H. Gould, N. G. Gunderson, J. B. Kelly, G. Pall, and the proposer.

Editorial Note. The proposer's solution is similar to the above, and the remaining solutions use different methods. Can some reader supply the desired reference?

Regular Polygons

4086 [1943, 391]. *Proposed by P. Erdös, Princeton, N. J.*

Let $A_1, A_2, \cdots, A_{2n+1}$ be the vertices of a regular polygon, and O any point in its interior. Show that at least one of the angles A_iOA_j satisfies the relation.

$$\pi\left(1-\frac{1}{2n+1}\right) \leqq A_iOA_j < \pi.$$

Editorial Note. R. K. Allen, L. M. Kelly, and C. R. Phelps state that the theorem of the problem is incorrectly stated as is shown by the regular pentagon. The point O is taken as the intersection of diagonals A_1A_3 and A_2A_4. The greatest angle $A_iOA_j<\pi$ is given by $A_5OA_3=126°$, which is less than the required 144°.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Dr. R. E. von Mises of Harvard University and Professor John von Neumann of the Institute for Advanced Study have been elected to membership in the American Academy of Arts and Sciences.

Professor Oscar Zariski of The Johns Hopkins University has been elected to membership in the National Academy of Sciences.

Dr. O. S. Adams of the U. S. Coast and Geodetic Survey has retired.

Dr. A. E. Basch of the College of the City of New York has been appointed to an assistant professorship in Mechanics at Rensselaer Polytechnic Institute.

Dr. Helen P. Beard of Newcomb College has been promoted to an assistant professorship.

Dr. R. A. Beaumont of the University of Washington has been promoted to an assistant professorship.

Assistant Professor E. M. Beesley of the University of Nevada has been promoted to an associate professorship. He is acting head of the department.

Assistant Professor J. R. Britton of the University of Colorado has been promoted to an associate professorship.

Dr. R. H. Bruck of the University of Wisconsin has been promoted to an assistant professorship.

Associate Professor J. W. Cell of North Carolina State College has been granted a leave of absence to serve as a mathematician at Aberdeen Proving Grounds.

Dr. Esther Comegys of the University of Maine has been promoted to an assistant professorship.

W. S. Dawkins has been appointed to a professorship at Loyola College, Baltimore, Maryland.

Assistant Professor L. A. Dye of The Citadel, Charleston, South Carolina, has been promoted to an associate professorship.

Dr. C. J. Everett of the University of Wisconsin has been promoted to an assistant professorship.

Dr. Mariano García of the University of Richmond has been appointed to an assistant professorship at the College of Agriculture and Mechanic Arts of the University of Puerto Rico.

Dr. Hilda P. Geiringer of Bryn Mawr College has been appointed to a professorship at Wheaton College, Norton, Massachusetts.

Professor G. A. Hedlund of the University of Virginia has been granted a leave of absence to serve with the applied mathematics group at Columbia University.

Assistant Professor S. A. Jennings of the University of British Columbia has been promoted to an associate professorship.

Dr. Wilfred Kaplan of the University of Michigan has been promoted to an assistant professorship.

Professor N. J. Lennes of Montana State University has retired.

Assistant Professor Norman Levinson of Massachusetts Institute of Technology has been promoted to an associate professorship.

Assistant Professor Eugene Lukacs of Illinois College has been appointed to an associate professorship at Berea College.

Assistant Professor M. S. Macphail of Acadia University has been promoted to an associate professorship.

Dr. C. T. McCormick of Fort Hays Kansas State College has been appointed to an associate professorship at the Illinois State Normal University.

Assistant Professor W. C. McDaniel of Southern Illinois Normal University has been promoted to an associate professorship.

Assistant Professor A. N. Milgram of the University of Notre Dame has been promoted to an associate professorship.

Assistant Professor David Moskovitz of the Carnegie Institute of Technology has been promoted to an associate professorship.

Assistant Professor D. C. Murdoch of the University of Saskatchewan has been appointed to an associate professorship at the University of British Columbia.

Assistant Professor Abba V. Newton of Smith College has been appointed to an assistant professorship at Vassar College.

Dr. K. L. Nielsen of Louisiana State University has been promoted to an assistant professorship. He is on leave with the Chance Vought Aircraft Research Division.

Assistant Professor H. L. Olson of Southwestern University, Georgetown, Texas, has been appointed head of the mathematics department at Western Union College, Le Mars, Iowa.

Assistant Professor E. L. Post of the College of the City of New York has been promoted to an associate professorship.

Associate Professor W. T. Reid of the University of Chicago has been appointed to a professorship at Northwestern University.

Associate Professor L. V. Robinson of Marshall College has been appointed to an associate professorship at Emory University, Georgia.

Assistant Professor W. J. Robinson of Allegheny College has been appointed to a professorship at Centre College of Kentucky.

Assistant Professor R. G. Sanger of the University of Chicago has been appointed assistant dean of students of the Division of the Physical Sciences.

Dr. Seymour Sherman of Curtiss Wright Corporation has been appointed research associate in the Allegany Ballistics Laboratory at Cumberland, Maryland.

Professor Gertrude Smith of Vassar College has retired with the title professor emeritus.

Dr. R. G. Sturm has been appointed to a professorship of engineering mechanics at Purdue University.

Assistant Professor E. G. Swafford of Fort Hays Kansas State College has been granted a leave of absence to accept a temporary appointment to an assistant professorship at Park College, Parkville, Missouri.

Assistant Professor E. C. Varnum of the University of Wyoming has been appointed mathematical physicist in the development laboratory of the Barber-Colman Company of Rockford, Illinois.

Assistant Professor J. F. Wardwell of Colgate University has been promoted to an associate professorship.

Professor W. M. Whyburn of the University of California at Los Angeles has been appointed to the presidency of Texas Technological College.

Dr. Frantisek Wolf of the University of California has been promoted to an assistant professorship.

Dr. Alexander Wundheiler of the College of the City of New York has been appointed to an assistant professorship in physics at the University of Rochester.

The following appointments to instructorships are announced:

Columbia University: P. B. Norman
Hollins College (instructor in education): Frances E. Falvey
Rice Institute: Dr. Albert Newhouse
Russell Sage College: Dr. Helene Reschovsky
Tulane University: Dr. R. L. Swain, Dr. Bernard Vinograde, Dr. Nelson Van de Luyster
Wellesley College: Ellen E. Fedder
Williams College: Dr. Abraham Seidenberg

Professor Emeritus F. G. Reynolds of the College of the City of New York died June 8, 1944. He was a charter member of the Mathematical Association.

WAR INFORMATION

EDITED BY C. V. NEWSOM

Send news reports upon the utilization of mathematicians or mathematics in war activities to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE ASBURY PARK PRE-V-12 PROGRAM

Previous to the term of the Navy V-12 Program starting July 1, 1944, approximately 2,500 men, who had been on active duty in the Navy, reported to the U. S. Naval Reserve Pre-Midshipmen's School at Asbury Park, New Jersey,

for refresher courses in mathematics, English, and physics. The Pre-V-12 curriculum was designed for a period of two months, and classes were to be held three times a week in each of the three subjects. In general, it was anticipated that there would be a minimum of two hours of preparation for each hour of class work. Many of the men were unable to arrive for the start of the program on May 1; in fact, the average period of refresher training proved to be about six weeks.

No grades were assigned, and no screening was done by the academic officers in charge of the program. The motivation was simply: "Here is an opportunity to learn, or re-learn, as much as you can before you go to college." The great majority of men responded well to this approach, and seemed to profit from the opportunity given them.

Virtually all men in the Pre-V-12 Program were given the C-2 and the C-3 tests developed by the Navy Test Research Unit at Princeton; these are the same tests which have been given to civilian applicants for V-12. The C-2 test was given to the men upon their arrival at Asbury Park, and the C-3 test was given at the end of the course of study. It is noteworthy that those trainees who were able to be present for virtually the entire program were able to improve their scores upon the mathematics sections of these tests by an average of 10.5 points; the total score possible in mathematics is 60 points. Even the large group of men arriving after May 17 made an average improvement of 8.1 points. On the basis of the showing made at Asbury Park, a report was drawn up for each trainee recommending V-12 placement in mathematics, English, and physics.

The program is to be repeated this fall previous to the term starting November 1. It is planned again to spend about two months upon refresher training; moreover, the general character of the curriculum will not be changed. Approximately 900 men have been assigned to Asbury Park for the second program.

Lieutenant Jackson B. Adkins, USNR, is head of the mathematics department of the Pre-Midshipmen's School at Asbury Park, and he organized the mathematics curriculum in the Pre-V-12 Program. All texts employed are reprints made for the United States Armed Forces Institute.

Early in the program, it is emphasized to trainees that the refresher course in mathematics has two major objectives, namely, "(1) to enable the student to acquire in an intelligent fashion the vocabulary of the subject, (2) to help the student to develop a skill in the technical processes that will be sufficient for the V-12 courses to build upon." The meaning of these phrases to the student is explained by the syllabus of the course in the following paragraphs.

"The first objective implies more than could be acquired by studying a dictionary of the terms. It means that the processes of the subject will be described in words so that an understanding of them can be acquired. By this procedure the student will not learn, for example, that $2x-3=5$ is solved by 'transposing' the 3 and dividing by 2, but will learn instead: 'Add 3 to both sides, then divide both sides of the equation by 2, the coefficient of the unknown term.' This

apparently cumbersome procedure is the only way to acquire an intelligent understanding that will be useful in successive lessons. Nothing more needs to be learned for more complicated problems. Nothing needs to be unlearned. The student will defeat himself unless he learns the words that describe the processes. The importance of 'talking' the subject cannot be overemphasized.

"Skill in the technical processes cannot be acquired until the processes are understood in the way indicated in the first objective. This means that the correct way to work on the technical processes is to 'make haste slowly.' Great skill and facility will not be acquired by some students in this brief course. But, if the course is properly studied, all students will be able to *do* any of the processes involved in the course. They may not be able to do them very rapidly, but they *will* be able to *do* them, and to get the right answers."

The course itself has been organized in twenty lessons, starting with simple arithmetic and ending with a brief study of the conic sections. A student assignment sheet of considerable detail has been developed for each lesson. Also, instructor's sheets pertaining to the various lessons were given to each Pre-V-12 instructor. An outline of the course is given below.

1. The Fundamental Operations with Decimal Fractions. How to read a decimal; place-value notation; sum; remainder; difference; product; multiplicand; multiplier; quotient; divisor; dividend; fundamental principle of fractions; multiples of 10; meaning of percent; rounding off; involution; evolution; square root table; order of operations; parentheses.

2. Common Fractions. Fundamental principle of fractions; meaning of fractions; prime number; factor; factoring; rules for reducing to lowest terms; adding, subtracting, dividing, multiplying; proper and improper fractions; mixed number; percent; ratio.

3. Algebraic Notation and Language. A study of $A=bh$; function; constant; variable; ratio; direct and inverse variation; locus; graphs; coordinates; axes; directed numbers; survey of operations with directed numbers.

4. The Fundamental Operations with Directed Numbers. Literal numbers; binomials; trinomials; polynomials; exponents; powers of signed numbers; special products; long division of polynomials.

5. Simple Equations. Rule for solving linear equations; literal equations; percentage problems.

6. Factoring. Term *vs* factor; a^2-b^2; $(a\pm b)^2-c^2$; ax^2+bx+c; long division; primes; perfect square and perfect cube factors.

7. Fractions. The fundamental operations; binomial denominators; the signs of a fraction; complex fractions.

8. Fractions. Fractional equations with binomial denominators; literal fractional equations; ratio; reciprocal.

9. Powers, Roots, and Radicals. Fundamental operations; binomial denominators; rationalizing the denominator; arithmetical square root.

10. Powers, Roots, and Radicals. Checking roots of $ax^2+bx+c=0$; negative exponents; inequalities.

11. Quadratic Equations. Completing the square; formula; factoring.
12. Equations. Final generalization of procedure.
13. Sets of Equations. Elimination by substitution.
14. The Right Triangle. Sum of angles; inscription in a circle; altitude on hypotenuse; Pythagorean Theorem; area; Hero's Formula.
15. The Measurement of Plane and Solid Figures. Polygons; circle; cylinder; cone; sphere; area; volume; variation; functions.
16. Numerical Trigonometry. Tables of natural functions.
17. Logarithms.
18. Trigonometric Tables.
19. The Oblique Triangle. Law of sines; law of cosines.
20. Conic Sections.

RECIRCULARIZATION OF REGISTRANTS IN THE NATIONAL ROSTER

The National Roster of Scientific and Specialized Personnel is now over four years old. From its inception, one of its procedures has been the continued recircularization of all registrants in its files in order to keep the file up to date with regard to addresses and added fields of specialization that any individual registrant may wish to indicate. The recent recircularization is a part of this continuing policy. Mathematicians not yet registered with the Roster, including recent graduates, should take steps to have their names included in the record.

Both the Army and the Navy have adopted the Roster's numerical coding system for describing the qualifications of officers and certain enlisted personnel in their Services. This fact, together with the continued vital demand for scientists in certain areas in connection with the war, make it seem extremely important that the records of the Roster be kept exactly up to date.

There are, of course, many thousands of individuals in the Armed Forces with professional and scientific competencies of a high order. The Roster has been requested to participate in the development of procedures which will assist these individuals in returning to civilian life when full demobilization begins. In this connection, the Roster is already working upon the problem of the most effective placement of returning service men who are registered with it.

THE SECTION ON EDUCATION IN THE G.I. BILL

To assist mathematicians who may be giving consideration to postwar plans, important parts of the education section of the G.I. Bill (S. 1767), as finally passed, are quoted below.

"Any person who served in the active military or naval service on or after September 16, 1940, and prior to the termination of the present war, and who shall have been discharged or released therefrom under conditions other than dishonorable, and whose education or training was impeded, delayed, interrupted, or interfered with by reason of his entrance into the service, or who desires a refresher or retraining course, and who either shall have served ninety days or more, exclusive of any period he was assigned for a course of education

or training under the Army specialized training program or the Navy college training program, which course was a continuation of his civilian course and was pursued to completion, or as a cadet or midshipman at one of the service academies, or shall have been discharged or released from active service by reason of an actual service-incurred injury or disability, shall be eligible for and entitled to receive education or training under this part: Provided, That such course shall be initiated not later than two years after either the date of his discharge or the termination of the present war, whichever is the later: Provided further, That no such education or training shall be afforded beyond seven years after the termination of the present war: And provided further, That any such person who was not over 25 years of age at the time he entered the service shall be deemed to have had his education or training impeded, delayed, interrupted, or interfered with.

"Any such eligible person shall be entitled to education or training, or a refresher or retraining course, at an approved educational or training institution, for a period of one year (or the equivalent thereof in continuous part-time study), or for such lesser time as may be required for the course of instruction chosen by him. Upon satisfactory completion of such course of education or training, according to the regularly prescribed standards and practices of the institutions, except a refresher or retraining course, such person shall be entitled to an additional period or periods of education or training, not to exceed the time such person was in the active service on or after September 16, 1940, and before the termination of the war, exclusive of any period he was assigned for a course of education or training under the Army specialized training program or the Navy college training program, which course was a continuation of his civilian course and was pursued to completion, or as a cadet or midshipman at one of the service academies, but in no event shall the total period of education or training exceed four years.

"Such person shall be eligible for and entitled to such course of education or training as he may elect, and at any approved educational or training institution at which he chooses to enroll whether or not located in the State in which he resides, which will accept or retain him as a student or trainee in any field or branch of knowledge which such institution finds him qualified to undertake or pursue: Provided, That, for reasons satisfactory to the Administrator, he may change a course of instruction: And provided further, That any such course of education or training may be discontinued at any time, if it is found by the Administrator that, according to the regularly prescribed standards and practices of the institution, the conduct or progress of such person is unsatisfactory.

"From time to time the Administrator shall secure from the appropriate agency of each State a list of the educational and training institutions (including industrial establishments), within such jurisdiction, which are qualified and equipped to furnish education or training (including apprenticeship and refresher or retraining training), which institutions, together with such additional ones as may be recognized and approved by the Administrator, shall be deemed

qualified and approved to furnish education or training to such persons as shall enroll under this part.

"The Administrator shall pay to the educational or training institution, for each person enrolled in full time or part time course of education or training, the customary cost of tuition, and such laboratory, library, health, infirmary, and other similar fees as are customarily charged, and may pay for books, supplies, equipment, and other necessary expenses, exclusive of board, lodging, other living expenses and travel, as are generally required for the successful pursuit and completion of the course by other students in the institution: Provided, That in no event shall such payments, with respect to any person, exceed $500 for an ordinary school year: And provided further, That if any such institution has no established tuition fee, or if its established tuition fee shall be found by the Administrator to be inadequate compensation to such institution for furnishing such education or training, he is authorized to provide for the payment, with respect to any such person, of such fair and reasonable compensation as will not exceed $500 for an ordinary school year.

"While enrolled in and pursuing a course under this part, such person, upon application to the Administrator, shall be paid a subsistence allowance of $50 per month, if without a dependent or dependents, or $75 per month, if he has a dependent or dependents, including regular holidays and leave not exceeding thirty days in a calendar year.

"No department, agency, or officer of the United States, in carrying out the provisions of this part, shall exercise any supervision or control, whatsoever, over any State educational agency, or State apprenticeship agency, or any educational or training institution: Provided, That nothing in this section shall be deemed to prevent any department, agency, or officer of the United States from exercising any supervision or control which such department, agency, or officer is authorized by existing provisions of law, to exercise over any Federal educational or training institution.

"The Administrator of Veterans' Affairs is authorized and empowered to administer this title, and, insofar as he deems practicable, shall utilize existing facilities and services of Federal and State departments and agencies on the basis of mutual agreements with them.

"The Administrator may arrange for educational and vocational guidance to persons eligible for education and training under this part. At such intervals as he deems necessary, he shall make available information respecting the need for general education and for trained personnel in the various crafts, trades, and professions: Provided, That facilities of other Federal agencies collecting such information shall be utilized to the extent he deems practicable.

"As used in this part, the term 'educational or training institutions' shall include all public or private elementary, secondary, and other schools furnishing education for adults, business schools and colleges, scientific and technical institutions, colleges, vocational schools, junior colleges, teachers colleges, normal schools, professional schools, universities, and other educational institutions."

THE ANNUAL MEETING OF THE NEBRASKA SECTION

The twenty-first meeting of the Nebraska Section of the Mathematical Association of America was held at the University of Nebraska in Lincoln, Nebraska, on Saturday, May 6, 1944. Professor W. A. Dwyer, Chairman of the Section, presided.

The attendance was twenty-one, including the following fifteen members of the Association: M. A. Basoco, A. K. Bettinger, W. C. Brenke, C. C. Camp, A. R. Congdon, H. M. Cox, W. A. Dwyer, J. M. Earl, M. G. Gaba, C. B. Gass, F. S. Harper, W. N. Huff, H. W. Linscheid, H. L. Rice, Lulu L. Runge.

At the business meeting the following officers were elected for the next year: Chairman, F. S. Harper, University of Nebraska; Secretary, Lulu L. Runge, University of Nebraska; Member of Executive Committee, W. A. Dwyer, Creighton University. It was voted to hold the next meeting at Lincoln, Nebraska, on a date to be announced later.

The following papers were presented:

1. *A class of orthogonal polynomials*, by Professor W. C. Brenke, University of Nebraska.

A class of polynomials $y_n(x)$ was considered as derived from the generating function $e^t f(xt)$, where $f(\theta)$ can be expanded as a formal power series in θ without missing terms. It was shown that, if the polynomials $y_n(x)$ form an orthogonal system, the reciprocals of the coefficients of θ in this power series form a "moment sequence." The result was extended to the more general type of generating function $\theta(t)f(xt)$.

2. *Particular instances of the uniform convergence of power series on an infinite interval*, by Professor J. M. Earl, Municipal University of Omaha.

Professor Earl remarked that if $f(x)$ is a given function of x whose power series expansion converges at each point of the infinite interval $(-\infty, \infty)$, if $S_n(x)$ is the nth partial sum of that series, and if n is fixed, then there usually exists an x such that the magnitude of the difference $f(x)-S_n(x)$ exceeds any preassigned positive number. On the other hand, if $r(x)$ is a certain non-negative weight function whose properties depend on those of $f(x)$, it can be shown that

$$r(x)\,|\,f(x) - S_n(x)\,| < \epsilon_n$$

uniformly on the interval $(-\infty, \infty)$, where ϵ_n depends on n but not on x, and tends toward zero as n becomes infinite. The first member of the inequality above is called the "weighted error."

Upper bounds ϵ_n for a few familiar functions and series were exhibited, and the order of the weighted uniform convergence was obtained. For example, if $r(x)=e^{-x^2}$, $f(x)=e^x$, and the series is the Maclaurin series for e^x, then the upper

bound ϵ_n for the weighted error on the infinite interval $(-\infty, \infty)$ is a constant multiple of e^{-n}.

3. *A theorem on unit groups of simple algebras*, by Professor Ralph Hull, University of Nebraska.

This paper was read by title.

4. *Problem in Diophantine analysis*, by Professor O. C. Collins, University of Nebraska, introduced by Professor W. A. Dwyer.

In this paper it was pointed out that a positive integer n can be represented in the form a^2-ab+b^2, where a and b are positive integers, only if $4n-3(a-b)^2$ is a perfect square. By consideration of $f(x+r)$, where $f(x)=x(x-a)(x-b)$, it was shown that the number of distinct representations is less than $\sqrt{n}/3$.

5. *A program of high school and college testing*, by Professor H. M. Cox, University of Nebraska.

The speaker stated that achievement examinations in mathematics and English were given this spring in one hundred and thirty Nebraska high schools to more than six thousand pupils. Ten colleges in Nebraska cooperated in the use of one or more guidance examinations administered as a part of the registration procedure for new students. The results of these examinations in high schools and colleges can be expressed in terms of *scaled scores* which represent approximately equal linear displacements in a normalized distribution. Thus the achievement of a student can be compared not only with the achievement of others in his own school and class, but also with the normative distribution composed of entering freshmen at the University of Nebraska.

6. *Correlation between machine grades and others*, by Professor C. C. Camp, University of Nebraska.

Professor Camp found that, for A.S.T.P. students in analytics, integral calculus, and algebra, the correlation between machine grades and others ranged from 0.41 to 0.72. Multiple choice tests were compared with each other, with final G. I. examinations, with class grades, or with term grades independent of such tests. In six of these comparisons 225 or more students were involved. In integral calculus, the grades of 145 students were studied. A limited number of civilian classes were also studied, and the mean of the correlation from quiz to quiz was 0.52. The longer the test and the more advanced the students, the higher the correlation was found to be.

7. *Panel discussion*: *Review of the army program*, by Professors W. C. Brenke, A. K. Bettinger, F. S. Harper, and A. R. Congdon.

8. *Panel discussion*: *Mathematics for returned service men*, by Professors M. G. Gaba and H. M. Cox.

9. *A photo-electric photometer of novel design*, by Professor O. C. Collins, University of Nebraska.

LULU L. RUNGE, *Secretary*

THE ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-seventh annual meeting of the Kentucky Section of the Mathematical Association of America was held at the University of Kentucky, Lexington, Kentucky, on Saturday, April 29, 1944, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor Charles Hatfield, Chairman of the Section, presided.

There were twenty in attendance, including the following eleven members of the Association: P. P. Boyd, M. C. Brown, H. H. Downing, Charles Hatfield, Aughtum S. Howard, W. R. Hutcherson, C. G. Latimer, F. Elizabeth LeStourgeon, Sallie E. Pence, D. E. South, H. A. Wright.

At the business meeting the following officers were elected for the coming year: Chairman, Aughtum S. Howard, Kentucky Wesleyan College; Secretary, M. C. Brown, University of Kentucky.

The following program was presented:

1. *Algebraic solutions of linear differential equations of the first order with linear coefficients*, by Professor H. H. Downing, University of Kentucky.

In this discussion the solutions of the differential equation

$$(ax + by + c)dx + (a'x + b'y + c')dy = 0$$

were obtained by the usual methods, and conditions in terms of the coefficients a, b, c, a', b', c' were found for which the solutions could be reduced to rational integral functions of x and y. Three cases were treated, namely: (1) $a/a' = b/b' = c/c'$; (2) $a/a' = b/b' \neq c/c'$; (3) $a/a' \neq b/b'$. Conditions in terms of the coefficients were found for which the algebraic solutions would reduce to quadratic form.

2. *The Tchebycheff inequality*, by Professor D. E. South, University of Kentucky.

Professor South discussed the proof of the Tchebycheff inequality for discrete and continuous variates. The Bernoulli limit theorem was then derived from the Tchebycheff criterion.

3. *Linkages*, by Dean P. P. Boyd, University of Kentucky.

Dean Boyd discussed the history of linkages from the time of Scheiner, the various linkages designed to approximate the drawing of a straight line, and the discoveries of Peaucellier and Hart. Emch's work in his *Kinematische Gelenksysteme* demonstrating the possibility of realizing a collineation by means of linkages was also described.

4. *Mathematics in the navy V-12 program*, by Professor W. R. Hutcherson, Berea College.

The speaker explained that every freshman in the program is required to take ten credit hours of mathematics. This includes algebra, trigonometry, and analytic geometry. Practically all sophomores (except premedics, predentals, and prechaplains) are required to take additional mathematics. Berea College

requires the weaker students to attend a remedial class in mathematics. Students, faculty, and navy officials have found that this extra work was justified by the results.

5. *Code writing*, by Professor Sallie E. Pence, University of Kentucky.

This paper dealt with the two principal methods of code writing, namely the substitutional and the transpositional methods. Examples of the four substitutional and three transpositional types were given, with some mention of the methods used in attempting to decipher them.

6. *Codes and matrices*, by Professor C. G. Latimer, University of Kentucky.

The speaker described certain ciphers based upon linear homogeneous transformations of n-tuples, with particular reference to involutory transformations. The connections between this paper and the preceding one were pointed out.

M. C. BROWN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-eighth Annual Meeting, Chicago, Ill., November 24–26, 1944.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA, Indianapolis, November 10, 1944
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, San Francisco, January 27, 1945
OHIO, Columbus, April 5, 1945
OKLAHOMA
PHILADELPHIA, Philadelphia, November, 1944
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Los Angeles, March 10, 1945
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1945

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